# ON EXACTNESS AND ISOPERIMETRIC PROFILES OF DISCRETE GROUPS

PIOTR W. NOWAK

ABSTRACT. We introduce a quasi-isometry invariant related to Property A and explore its connections to various other invariants of finitely generated groups. This allows to establish a direct relation between asymptotic dimension on one hand and isoperimetry and random walks on the other. We apply these results to reprove sharp estimates on isoperimetric profiles of some groups and to answer some questions in dimension theory.

A geometric characterization of amenable groups says that amenability is equivalent to existence of sequence of Følner sets in the group. The degree of "how amenable" the group is can be measured by the growth rate of Følner sets for a fixed set of generators - this is the so called *Følner function* of an amenable group, it was introduced by Vershik in the 70's (see Section 1 for definitions).

On the other hand one can study after Gromov the *isoperimetric profile* of any discrete group, i.e. a quantitative dependance between the volumes of a set $A$ and its boundary $\partial A$. It is a consequence of Følner's characterization of amenability that in the case of amenable groups finding the asymptotics of the Følner function is equivalent to finding those of the isoperimetric profile. These functions also turned out to play a role in probability theory on groups after they were linked with random walks and the decay of the heat kernel by Varopoulos.

In [Yu$_2$] Guoliang Yu introduced Property A, a weak, metric amenability condition. In this article we introduce a function $\mathrm{A}_X : \mathbb{N} \to \mathbb{N}$ which is a weak version of Vershik's Følner function. It is well defined for every discrete, locally finite metric space $X$ with Property A and it measures, roughly speaking, how well is Property A satisfied for the given space. If $X = \Gamma$ is a finitely generated group equipped with a word length metric (this is the situation which interest us the most) then a theorem due to Guentner, Kaminker and Ozawa [GK$_1$], [Oz] says that $\Gamma$ has Property A if and only if the reduced group $C^*$-algebra $C^*_r(\Gamma)$ is exact, thus the function $\mathrm{A}_\Gamma$ can be also interpreted as a measure of exactness of a discrete group.

We gain several benefits from introducing the function A. First, it is a quasi-isometry invariant. Second, it has the advantage over the classical Følner function that, just as Property A, it is well defined for much larger class of groups, which includes all $\delta$-hyperbolic groups, free products and many other non-amenable examples.

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The main feature of $A_\Gamma$ is that it is connected to other asymptotic invariants and can be estimated in many ways. And so after going over the preliminaries, definitions and basic properties in the first three sections, in Section 4 we show a direct link between Vershik's classical Følner function and our weak Følner function. This connection is quite natural, after all Property A is modeled after amenability. Formally our estimate is obtained through an averaging argument and allows to exhibit examples of groups with weak Følner functions growing arbitrarily fast. It also follows that if Thompson's group $F$ would turn out to have Property A then $A_F$ must grow faster than any polynomial.

In [Gro$_2$] Gromov introduced the notion of asymptotic dimension, a large-scale version of topological covering dimension. In section 5 we employ the fact that finite asymptotic dimension implies Property A to estimate $A_\Gamma$. More precisely, a metric space with finite asymptotic dimension has an associated *type function*, also defined in [Gro$_2$] as the second quasi-isometry invariant arising from the definition of asymptotic dimension. For spaces with finite asymptotic dimension the estimates we obtain on the weak Følner function are in terms of this type function. It follows in particular that $\delta$-hyperbolic groups have linear $A_\Gamma$.

It should be visible already from the two paragraphs above that $A_\Gamma$ combines the asymptotics arising in amenability with those arising in large-scale dimension theory. This relation between asymptotic dimension and isoperimetry, formulated rigorously in Theorem 6.1, is our main application of the function $A_\Gamma$. Consequently this also establishes a connection between asymptotic dimension and the decay of the heat kernel. The latter is one of the central themes in the study of random walks on finitely generated groups, we refer the reader to the articles [PS-C$_1$], [S-C$_1$] and [CS-CV] for a comprehensive overview of this subject.

These observations allows us to obtain results in different directions. As an example of an application to dimension theory, in Section 8 we construct for every $k = 1, 2, 3, \dots$ infinitely many finitely generated groups with asymptotic dimension equal to $k$ and infinite Assouad-Nagata dimension. In particular, this class contains the lamplighter groups. It also follows that these groups don't behave very well under coarse embeddings into finite products of trees, this contrasts with the case of hyperbolic spaces and groups, which embed quasi-isometrically into appropriately chosen products of this type ([BS$_2$], [BS$_3$]). We also show that our results can give sharp estimates on isoperimetric profiles, this is done in Section 7 by reproving, using asymptotic dimension methods, these sharp estimates for some examples.

Finally in the last section we briefly discuss possible further applications of $A_\Gamma$, such as the one to the Hilbert compression of metric spaces (see [GK$_2$]) and we list some questions that remain open.

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## 1. Necessary preliminaries

The definitions and some of the results work in a general setting of metric spaces, but we will mostly specialize to finitely generated groups with word length metric.

**Asymptotics.** We will usually think of a function $f : \mathbb{N} \to \mathbb{N}$ as a piecewise linear function $f : \mathbb{R} \to \mathbb{R}$, determined by it's values on the integers. We don't lose any information this way since all the functions considered in this article are nondecreasing.

Given two functions $f, g : \mathbb{N} \to \mathbb{N}$ we write $f \preceq g$ if $f(n) \leq C_1 g(C_2 n)$ for some constants $C_1, C_2$, and $f \sim g$ if $f \preceq g$ and $g \preceq f$. We will write $f \prec g$ if the inequality is strict, i.e. $f \preceq g$ but it is not true that $f \sim g$. We will also often say that $f$ is linear if $f(n) \preceq n$, polynomial if $f(n) \preceq n^k$ for some $k \in \mathbb{N}$ and so on.

We will sometimes write the inverse $f^{-1}$ of a function for which it is not clear if it has an actual inverse. What we mean by this is the inverse of a invertible function that has the same asymptotics as $f$.

**Discrete metric spaces.** We are going to consider discrete metric spaces which are geodesic on the large scale.

**Definition 1.1.** *A metric space is uniformly quasi-geodesic if there exist constants $C, L > 0$ such that for any $x, y \in X$ there exists a sequence $x = x_1, x_2 \ldots, x_{n-1}, x_n = y$ of points in $X$ such that $n$ depends only on $d(x, y)$ and*

$$\sum_{i=1}^{\infty} d(x_i, x_{i+1}) \leq C d(x, y)$$

*where $x_1 = x$, $x_n = y$ and $d(x_i, x_{i+1}) \leq L$.*

A discrete metric space $X$ has *bounded geometry* if for every $R > 0$ there exists a number $\mathcal{N}(R) > 0$ such that for every $x \in X$,

$$\#B(x, R) \leq \mathcal{N}(R)$$

holds. Such a space is necessarily countable.

All the metric spaces under consideration are assumed to have integer-valued metric, to be of bounded geometry and uniformly quasi-geodesic. For convenience we assume that quasi-geodesic condition is satisfied with with $C = L = 1$, just like in the case of finitely generated groups with the word length metric. All these conditions are not very restrictive.

**Volume growth.** We will always assume that our finitely generated groups are equipped with a proper word length metric obtained from a finite, symmetric generating set and we will denote by $|\cdot|$ the length function, $|\gamma| = d(\gamma, e)$. We will use the notation $B(r) = B(e, r) = \{ g \in \Gamma : |g| \leq r \}$ for the ball of radius $r$ in $\Gamma$. The symbol $\rho_\Gamma$ will be reserved for volume growth of $\Gamma$:

$$\rho_\Gamma(n) = \#B(n).$$

Growth of any finitely generated group is either exponential ($\rho_\Gamma \sim e^n$) or subexponential.

**Amenability.** Let $X$ be a set. Define

$$\ell_1(X)_{1,+} = \{ f \in \ell_1(X) \ : \ f \geq 0, \|f\|_1 = 1 \}.$$

In other words, $\ell_1(X)_{1,+}$ is the space of positive probability measures on $X$. Given a map $\xi : X \to \ell_1(X)_{1,+}$ where $x \mapsto \xi_x$, we denote by $\xi_x(y)$ the $y$-coordinate of the vector $\xi_x$.

For a function $f : \Gamma \to \mathbb{R}$ and $\gamma \in \Gamma$ we use the standard notation

$$(\gamma \cdot f)(x) = f(\gamma^{-1} x)$$

for all $x \in \Gamma$. We recall below the standard and useful in our case characterizations of amenability, general references on this topic are [BHV, Appendix G], [Pi], [Gre].

**Definition 1.2.** *A finitely generated group $\Gamma$ is amenable if any of the following equivalent conditions is satisfied:*

(i) ***(Invariant Mean Condition)*** *There exists a left invariant mean on $\ell_\infty(\Gamma)$, i.e. a positive, linear functional $\int \cdot\, dg$ on $\ell_\infty(\Gamma)$ such that $\int 1\, dg = 1$ and $\int \gamma \cdot f\, dg = \int f\, dg$ for any $\gamma \in \Gamma$;*

(ii) ***(Approximation by finitely supported measures)*** *For every $\varepsilon > 0$ and $R < \infty$ there exists a function $f \in \ell_1(\Gamma)_{1,+}$ such that*

$$\|\gamma \cdot f - f\|_{\ell_1(X)} \leq \varepsilon$$

*for all $|\gamma| \leq R$ and $\#\operatorname{supp} f < \infty$.*

**Følner functions and isoperimetric profiles.** The function describing growth of Følner sets was introduced by Vershik [Ver] and later studied in e.g. [KV], [GŻ], [Er]. The definition is the following:

**Definition 1.3** (**Følner function**). *For an amenable group $\Gamma$ define the function $\mathrm{Føl}_\Gamma : \mathbb{N} \to \mathbb{N}$,*

$$\mathrm{Føl}_\Gamma(n) = \min \#\operatorname{supp} f, \tag{1}$$

*with the minimum taken over all $f \in \ell_1(\Gamma)_{1,+}$ satisfying condition (ii) in Definition 1.2 with $\varepsilon = \frac{1}{n}$ and $R = 1$.*

The Følner function is usually defined in terms of growth of volumes of Følner sets, see e.g. [Er]. We leave it to the reader to check that these two definitions are equivalent when it comes to asymptotics, see e.g. the exposition in [BHV, Appendix G].

The isoperimetric profile of a finitely generated group $\Gamma$ is the function $J_\Gamma : \mathbb{N} \to \mathbb{N}$ defined in the following way:

$$J_\Gamma(n) = \sup_{\#A \leq k} \frac{\#A}{\#\partial A}.$$

The exact values of $\mathrm{Føl}_\Gamma$ and $J_\Gamma$ of course depend on the metric, the asymptotics however do not and are quasi-isometry invariants, thus we in particular omit the reference to the generating set. The advantage of $J_\Gamma$ over the Følner function is that it is well defined for every finitely generated group, while $\text{Føl}_\Gamma$ only for amenable ones. In the latter case these functions carry the same information, see [S-C$_2$] for a formula giving direct relation.

It is worth noting that the functions $J$ and Føl of quotient groups of $\Gamma$ give information on the large-scale isoperimetry on regular covers of Riemannian manifolds with fundamental group $\pi_1(\mathcal{M}) = \Gamma$, we refer the reader to Saloff-Coste's survey [S-C$_2$] and the references therein for more details.

Another important direction, as mentioned in the introduction, is the connection between isoperimetric profiles and random walks on groups, we again will not discuss these in detail here and we direct the reader to the articles [PS-C$_1$] and [S-C$_1$] which provide the necessary background.

## 2. Property A and all that

Property A was introduced by Yu in [Yu$_2$], we will use the definition given by Higson and Roe [HR].

**Definition 2.1.** *A discrete metric space has property A if for every $R < \infty$ and $\varepsilon > 0$ there exists a map $\xi : X \to \ell_1(X)_{1,+}$ and a number $S < \infty$ such that*

(i) $\|\xi_x - \xi_y\|_{\ell_1(X)} \le \varepsilon$ *whenever* $d(x,y) \le R$,
(ii) $\operatorname{supp} \xi_x \subseteq B(x,S)$ *for every* $x \in X$.

For example, amenable groups have Property A - in Definition 2.1 the function $\xi$ is given simply by the formula $\xi_\gamma = \gamma \cdot f$ for an appropriately chosen $f$ from condition (ii) in Definition 1.2.

Other classes of groups with Property A include free and more generally hyperbolic groups [Yu$_2$], Coxeter groups [DJ], linear groups [GHW], one-relator groups [Gu] and many more. In fact the only known groups which don't have Property A are the random groups constructed by Gromov [Gro$_3$], containing expanders in their Cayley graphs. It is also unknown whether Thompson's group $F$ has Property A (see e.g.[AGS]).

**Definition 2.2** ([Gro$_2$, 7.E.])**.** *Let $X, Y$ be metric spaces. A function $f : X \to Y$ is a coarse embedding if there exist non-decreasing functions $\varphi_-, \varphi_+ : [0,\infty) \to [0,\infty)$ satisfying*

(i) $\varphi_-(d_X(x,y)) \le d_Y(f(x), f(y)) \le \varphi_+(d_X(x,y))$ *for all* $x, y \in X$,
(ii) $\lim_{t\to\infty} \varphi_-(t) = +\infty$.

Note that for quasi-geodesic metric spaces $\varphi_+$ can always be chosen to be affine. If in addition $\varphi_-$ also can be chosen to be affine then $f$ is called a *quasi-isometric* embedding, and if for some constant $C > 0$ the image is $C$-dense in $Y$ then the embedding is in fact a coarse equivalence.

Property A was introduced mainly for the purpose of the following

**Theorem 2.3** ([Yu$_2$])**.** *If a metric space $X$ has Property A then $X$ admits a coarse embedding into the Hilbert space.*

This, on the other hand, has applications to the Novikov Conjecture through the following remarkable result.

**Theorem 2.4** ([Yu$_2$])**.** *If a bounded geometry metric space $X$ admits a coarse embedding into a Hilbert space then the coarse index map*

$$\mu_c : \lim_{d\to\infty} K_*(P_d(X)) \to K_*(C^*(X))$$

*is an isomorphism. In particular, if $X = \Gamma$ is a finitely generated group with the word length metric then the Novikov Conjecture is true for $\Gamma$.*

In the above statement $P_d(X)$ is the Vietoris-Rips complex of the space $X$ and $C^*(X)$ is the Roe algebra associated to $X$. See e.g. [Yu$_2$] and the references there for more on Coarse Baum-Connes Conjecture and applications to problems in geometry.

Also, by the work of Guentner, Kaminker and Ozawa (see [GK$_1$] and [Oz]) Property A for a finitely generated group is equivalent to exactness of the reduced group $C^*$-algebra $C^*_r(\Gamma)$. The reader will find much more information on Property A and coarse geometry in Roe's book [Roe$_1$].

## 3. Weak Følner functions associated to Property A

Let us first define some necessary notions.

**Definition 3.1.** *Let $X$ be a discrete metric space.*

**(A)** *For a map $\xi : X \to \ell_1(X)_{1,+}$ satisfying condition (i) in Definition 2.1 with $\varepsilon > 0$ and $R < \infty$ denote*

$$\mathcal{S}_X(\xi, R, \varepsilon) = \inf S,$$

$\mathcal{S}_X(\xi, R, \varepsilon) \in \mathbb{N} \cup \{\infty\}$, *where the infimum is taken over all $S > 0$ satisfying* $\operatorname{supp} \xi_x \subseteq B(x, S)$ *for every $x \in X$.*

**(B)** *Define*

$$\operatorname{rad}_X(R, \varepsilon) = \inf \mathcal{S}_X(\xi, R, \varepsilon),$$

$\operatorname{rad}(R, \varepsilon) \in \mathbb{N} \cup \{\infty\}$, *where the infimum is taken over all maps satisfying the conditions in (A) for $R$ and $\varepsilon$.*

**(C)** *If $\Gamma$ is a finitely generated group then for given $R < \infty$, $\varepsilon > 0$ by* $\operatorname{rad}^{eqv}_\Gamma(R, \varepsilon) \in \mathbb{N} \cup \{\infty\}$ *we denote the smallest $S$ for which there exists a function $f \in \ell_1(\Gamma)_{1,+}$ with* $\operatorname{supp} f \subseteq B(S)$ *satisfying condition (ii) in Definition 1.2 for all $\gamma \in \Gamma$ such that $|\gamma| \leq R$.*

In other words, $\operatorname{rad}^{eqv}_\Gamma$ is the notion resulting from restricting (A) and (B) to considering only functions $\xi : \Gamma \to \ell_1(\Gamma)_{1,+}$ given by translates of a single function $f \in \ell_1(\Gamma)_{1,+}$, i.e. $\xi(\gamma) = \gamma \cdot f$ for every $\gamma \in \Gamma$ and for some fixed, appropriately chosen $f \in \ell_1(X)_{1,+}$.

The following is the main definition in this article.

**Definition 3.2** (**Weak Følner function of a metric space** $X$)**.** *Let $X$ be a metric space with Property A. Define the function $\mathrm{A}_X : \mathbb{N} \to \mathbb{N}$ by the formula*

$$\mathrm{A}_X(n) = \mathrm{rad}_X\left(1, \frac{1}{n}\right).$$

Clearly the function $\mathrm{A}_X$ is well-defined and non-decreasing. We will be interested in estimating the asymptotic behavior of $\mathrm{A}_X$. This does not depend on the choice of $R = 1$ and the sequence $\frac{1}{n}$ up to constants, the argument will be given further in this section.

**Example 3.3.** Let $X$ be a bounded metric space. Then $\mathrm{A}_X \sim \mathrm{const}$. In fact, $\mathrm{A}_X(n) = \mathrm{diam}\, X$ for all $n$ large enough.

**Example 3.4.** Let $T$ be any locally finite tree. Then $\mathrm{A}_T \preceq n$. Indeed, recall from [$\mathrm{Yu}_2$] that for a fixed $R < \infty$ and $\varepsilon > 0$ Property A for the tree is constructed by fixing a point $\omega$ on the boundary of $T$ and taking normalized characteristic functions of the geodesic segments of length $\frac{2R}{\varepsilon}$ on the geodesic ray starting from $x$ in the direction of $\omega$.

**Example 3.5.** Let $\Gamma$ be a finitely generated group with polynomial growth. Then $\mathrm{A}_\Gamma \preceq n$. In this case the normalized characteristic functions of balls of radius $n$ do the job.

We now move on to prove the most natural properties - estimate for subspaces, direct products and invariance under quasi-isometries. For the first one, we will use the fact that Property A is hereditary [Tu].

**Proposition 3.6.** *Let $Y$ have Property A and $X \subseteq Y$. Then $X$ has Property A and for any $R < \infty$, $\varepsilon > 0$*

$$\mathrm{rad}_X(R, \varepsilon) \leq 3\,\mathrm{rad}_Y(R, \varepsilon)$$

*Proof.* For every $y \in Y$ let $p(y) \in X$ be a point such that $d(y, p(y)) \leq 2d(y, X)$. Define an isometry $\mathcal{I} : \ell_1(Y) \to \ell_1(X \times Y)$ by the formula

$$\mathcal{I} f(x, y) = \begin{cases} f(y) & \text{if } x = p(y) \\ 0 & \text{otherwise} \end{cases}$$

Let $\varepsilon > 0$ and $R < \infty$. By definition of Property A there exist a number $S < \infty$ and a map $\xi : Y \to \ell_1(Y)$ such that $\|\xi_y - \xi_{y'}\|_{\ell_1(Y)} \leq \varepsilon$ if $d(y, y') \leq R$ and $\mathrm{supp}\,\xi_y \subseteq B(y, S)$ for every $y \in Y$. Define $\widetilde{\xi} : X \to \ell_1(X)_{1,+}$ by the formula

$$\widetilde{\xi}_x(z) = \sum_{y \in Y} \mathcal{I}\xi_x(z, y).$$

Then it is easy to check that

$$\|\widetilde{\xi}_x - \widetilde{\xi}_{x'}\|_{\ell_1(X)} \leq \varepsilon$$

whenever $d(x, x') \leq R$ and

$$\mathrm{supp}\,\xi_x \subseteq B(x, 3S).$$

∎

A direct consequence is the following.

**Proposition 3.7.** *Let $X \subseteq Y$ be a subspace. Then* $\mathrm{A}_X \leq \mathrm{A}_Y$.

**Direct products.** We consider direct products with the $\ell_1$-metric: if $X_1, X_2$ are metric spaces with metrics $d_1$ and $d_2$ respectively then the metric on the direct product is given by the formula

$$d(x,y) = d_1(x_1,y_1) + d_2(x_2,y_2),$$

where $x = (x_1,x_2) \in X_1 \times X_2$, $y = (y_1,y_2) \in X_1 \times X_2$.

**Proposition 3.8.** *Let $X_1$, $X_2$ be countable discrete metric spaces with property A. Then*

$$\mathrm{A}_{X_1 \times X_2} \sim \max\left(\mathrm{A}_{X_1}, \mathrm{A}_{X_2}\right).$$

*Proof.* Let $R = 1$, $\varepsilon > 0$ and let the maps $\xi : X \to \ell_1(X)_{1,+}$, $\eta : Y \to \ell_1(Y)_{1,+}$ realize Property A for $R = 1$ and $\varepsilon$, for $X$ and $Y$ and respectively, with diameters of the supports $S_X$ and $S_Y$ respectively. Then the map $\xi \otimes \eta : X \times Y \to \ell_1(X \times Y)_{1,+}$ defined by

$$\xi \otimes \eta_{(x,y)}(z,w) \quad = \quad \xi_x(z)\,\eta_y(w),$$

satisfies

$$\operatorname{supp}\left(\xi \otimes \eta_{(x,y)}\right) \subseteq B((x,y), S_X + S_Y) \subseteq B((x,y), 2\max(S_X,S_Y)).$$

For $R = 1$ we also have the following estimate:

$$\begin{aligned}
\|\xi \otimes \eta_{(x,y)} - \xi \otimes \eta_{(x',y')}\|_{\ell_1(X \times Y)} &= \sum_{z \in X, w \in Y} |\xi_x(z)\eta_y(w) - \xi_{x'}(z)\eta_{y'}(w)| \\
&\leq \sum_{z \in X, w \in Y} |\xi_x(z)\eta_y(w) - \xi_x(z)\eta_{y'}(w)| \\
&\qquad + \sum_{z \in X, w \in Y} |\xi_x(z)\eta_{y'}(w) - \xi_{x'}(z)\eta_{y'}(w)| \\
&\leq \|\xi_x - \xi_{x'}\|_{\ell_1(X)} + \|\eta_y - \eta_{y'}\|_{\ell_1(Y)} \leq \varepsilon.
\end{aligned}$$

The last inequality follows from the fact that since $d((x,y)(x',y')) = R = 1$ then either $x = x'$ or $y = y'$. This proves $\mathrm{A}_{X_1 \times X_2} \leq \max\left(\mathrm{A}_{X_1}, \mathrm{A}_{X_2}\right)$. The estimate "$\geq$" follows from Proposition 3.7. □

Permanence properties of groups with Property A have been extensively studied in connection to the Novikov Conjecture, see e.g. [Be], [DG], [CDGY], [Tu], so estimates of this sort are possible also for e.g. free products, extensions, some direct limits, groups acting on metric spaces etc. It would be interesting to obtain sharp estimates for $\mathrm{A}_\Gamma$ of groups resulting from such constructions.

**Invariance under quasi-isometries.** We devote the rest of this section to proving large-scale invariance of the asymptotics of $\mathrm{A}_X$, we will in particular estimate how does the weak Følner function behave under coarse equivalences that are not necessarily quasi-isometries. Strictly for that purpose for $\kappa, R \in \mathbb{N}$ define the function $\mathrm{A}_X^{\kappa,R}(n) = \mathrm{rad}_X(R, \frac{\kappa}{n})$. With this definition $\mathrm{A}_X = \mathrm{A}_X^{1,1}$.

**Lemma 3.9.** *For a fixed $R < \infty$ and $\kappa \in \mathbb{N}$ we have*

$$\mathrm{A}_X^{1,R} \ \sim \ \mathrm{A}_X^{\kappa,R}. \ \square$$

**Lemma 3.10.** *Let $X$ have Property A. Then for any $R, R' \in \mathbb{N}$ we have*

$$\mathrm{A}_X^{1,R} \sim \mathrm{A}_X^{1,R'}.$$

*Proof.* If $R \le R'$ then obviously $\mathrm{rad}_X(R,\varepsilon) \le \mathrm{rad}_X(R',\varepsilon)$ for any $\varepsilon$ and the inequality "$\le$" follows. Conversely, assume that $R' \le R$. If $d(x,y) \le R$ and that we're given the function $\xi$ from the definition of Property A for $R'$ and $\varepsilon$. Then by the uniform quasi-geodesic condition on $X$ (Definition 1.1) with $\kappa$ equal to the largest integer smaller than $R/R'$, we have

$$\|\xi_x - \xi_y\|_{\ell_1(X)} \ \le \ \sum_{i=0}^{\kappa-1} \|\xi_{x_i} - \xi_{x_{i+1}}\|_{\ell_1(X)} \ \le \ \kappa\varepsilon_n,$$

where the $x = x_0, x_1, ..., x_{k-1}, x_\kappa = y$ are such that $d(x,y) \le \sum_{i=0}^{\kappa} d(x_i, x_{i+1})$ and $d(x_i, x_{i+1}) \le R'$. This gives the inequality $\mathcal{S}_X(\xi, R', \varepsilon) \le \mathcal{S}_X(\xi, R, \kappa\varepsilon)$, and consequently

$$\mathrm{rad}_X(R', \varepsilon) \le \mathrm{rad}_X(R, \kappa\varepsilon).$$

This together with the previous lemma proves the assertion. $\square$

Having proved that the asymptotics of $\mathrm{A}_X^{\kappa,R}$ depend neither on $R$ nor on $\kappa$, as a consequence we get the desired statement on large-scale behavior of $\mathrm{A}_X$.

**Theorem 3.11.** *Let $X, Y$ be metric spaces and let $Y$ have Property A. Let $f : X \to Y$ be a coarse embedding. Then $X$ has Property A and*

$$\mathrm{A}_X \preceq \varphi_-^{-1} \circ \mathrm{A}_Y.$$

*In particular, if $X$ and $Y$ are quasi-isometric then* $\mathrm{A}_X \sim \mathrm{A}_Y$.

*Proof.* Let $f : X \to Y$ be the coarse embedding with Lipschitz constant $L$ and distortion $\varphi_-$. Since we're only interested in the asymptotic behavior, we may assume that for large $t \in R$, $\varphi_-(t)$ is strictly increasing. Also by Proposition 3.6 without loss of generality we may assume that $f$ is onto.

For every point $y \in Y$ choose a unique point $x_y$ in the preimage $f^{-1}(y)$. This gives an inclusion $\ell_1(Y)_{1,+} \subseteq \ell_1(X)_{1,+}$. Since $Y$ has Property A, for every $\varepsilon > 0$ and $R < 0$ there exists a map $\xi : Y \to \ell_1(Y)_{1,+}$ and a number $S > 0$ satisfying conditions from Proposition 2.1. Choose $R$ large enough so that $\varphi_-(R) \ge 1$ and define a map $\eta : X \to \ell_1(Y)_{1,+} \subseteq \ell_1(X)_{1,+}$ setting

$$\eta_x(z) = \begin{cases} \xi_{f(x)}(y) & \text{if } z = x_y, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that $\varphi$ satisfies the required conditions and that

$$\mathcal{S}_X(R,\varepsilon,\eta) \le \varphi_-^{-1}(\mathcal{S}_Y(LR,\varepsilon,\xi)).$$

This, with Lemma 3.9 gives

$$\mathrm{A}_X \;\sim\; \mathrm{A}_X^{1,R} \;\;\le\;\; \varphi_-^{-1}\circ \mathrm{A}_Y^{1,LR} \;\;\sim\;\; \varphi_-^{-1}\circ \mathrm{A}_Y.$$

□

**Remark 3.12.** One can also define another invariant considering the volume of the supports of functions appearing in the definition of Property A, or volumes of the sets appearing in the original definition of Property A (see [Yu$_2$]). By the bounded geometry condition in the first case and the argument in [HR, Lemma 3.5] in the second, these functions are well-defined for discrete groups and they are quasi-isometry invariants. This would be closer to the definition of the isoperimetric profile, however one might run into problems trying to average such invariants in the case of amenable groups (see the next section).

## 4. Relation to Følner functions

In this section we will show that A is indeed a weak version of the function $\mathrm{Føl}_\Gamma$ . In order to do this we need to directly relate the numbers rad and rad$^{eqv}$, this is done in this next Theorem, which was proved by the author in [No$_2$] for the purpose of distinguishing Property A and coarse embeddability into the Hilbert space. We recall it together with the proof.

**Theorem 4.1** ([No$_2$])**.** *Let $\Gamma$ be finitely generated amenable group, $R \ge 1$ and $\varepsilon > 0$. Then*

$$\mathrm{rad}_\Gamma(R,\varepsilon) \quad = \quad \mathrm{rad}_\Gamma^{eqv}(R,\varepsilon)\,.$$

*Proof.* To show the inequality $\mathrm{rad}_\Gamma(R,\varepsilon) \le \mathrm{rad}_\Gamma^{eqv}(R,\varepsilon)$, given a finitely supported function $f \in \ell_1(\Gamma)_{1,+}$ satisfying condition (2) from Definition 1.2 for $R > 0$ and $\varepsilon > 0$ and all $\gamma \in \Gamma$ such that $|g| \le R$, consider the map $\xi : \Gamma \to \ell_1(\Gamma)_{1,+}$ defined by $\xi(\gamma) = \gamma \cdot f$.

To prove the other inequality assume that $\Gamma$ satisfies conditions from (2) of Proposition 2.1 for $R < \infty$, $\varepsilon > 0$ with $S > 0$ realized by the function $\xi : \Gamma \to \ell_1(\Gamma)_{1,+}$. For every $\gamma \in \Gamma$ define

$$f(\gamma) = \int_\Gamma \xi(g)(\gamma^{-1}g)\, dg.$$

This gives a well-defined function $f : \Gamma \to \mathbb{R}$, $\xi(g)(\gamma^{-1}g)$ as a function of $g$ belongs to $\ell_\infty(\Gamma)$ since $\xi(g)(\gamma) \le 1$ for all $\gamma, g \in \Gamma$.

First observe that if $|\gamma| > S$ then $\xi(g)(\gamma^{-1}g) = 0$ for all $g \in \Gamma$, thus $f(\gamma) = 0$ whenever $|\gamma| > S$. Consequently,

$$\begin{aligned} \|f\|_{\ell_1(\Gamma)} &= \sum_{\gamma\in B(S)} f(\gamma) = \sum_{\gamma\in B(S)} \int_\Gamma \xi(g)(\gamma^{-1}g)\, dg \\ &= \int_\Gamma \left( \sum_{\gamma\in B(S)} \xi(g)(\gamma^{-1}g) \right) dg = \int_\Gamma 1\, dg = 1. \end{aligned}$$

Thus $f$ is an element of $\ell_1(\Gamma)_{1,+}$.

If $\lambda \in \Gamma$ is such that $|\lambda| \le R$ then

$$\begin{aligned} \|f - \lambda\cdot f\|_{\ell_1(\Gamma)} &= \sum_{\gamma\in\Gamma} |f(\gamma) - f(\lambda^{-1}\gamma)| \\ &= \sum_{\gamma\in B(S)\cup\lambda B(S)} \left| \int_\Gamma \xi(g)(\gamma^{-1}g)\, dg - \int_\Gamma \xi(g)((\lambda^{-1}\gamma)^{-1}g)\, dg \right| \\ &= \sum_{\gamma\in B(S)\cup\lambda B(S)} \left| \int_\Gamma \xi(g)(\gamma^{-1}g)\, dg - \int_\Gamma \xi(\lambda^{-1}g)(\gamma^{-1}g)\, dg \right| \\ &= \sum_{\gamma\in B(S)\cup\lambda B(S)} \left| \int_\Gamma (\xi(g)(\gamma^{-1}g) - \xi(\lambda^{-1}g)(\gamma^{-1}g))\, dg \right| \\ &\le \int_\Gamma \left( \sum_{\gamma\in B(S)\cup\lambda B(S)} |\xi(g)(\gamma^{-1}g) - \xi(\lambda^{-1}g)(\gamma^{-1}g)| \right) dg \\ &\le \int_\Gamma \varepsilon\, dg = \varepsilon, \end{aligned}$$

since

$$\begin{aligned} \int_\Gamma \xi(g)((\lambda^{-1}\gamma)g)\, dg &= \int_\Gamma \lambda\cdot\left(\xi(g)((\lambda^{-1}\gamma)^{-1}g)\right) dg \\ &= \int_\Gamma \xi(\lambda^{-1}g)(\gamma^{-1}g)\, dg, \end{aligned}$$

this is a consequence of the invariance of the mean.

Thus for the previously chosen $R$ and $\varepsilon$ we have constructed a function $f \in \ell_1(\Gamma)_{1,+}$ satisfying $\|f - \gamma\cdot f\|_{\ell_1(\Gamma)} \le \varepsilon$ whenever $1 \le |\gamma| \le R$ and $\operatorname{supp} f \subseteq B(S)$ for the same $S$ as for $\xi$. This proves the second inequality. □

Since the Følner function measures the volume of the support of a function and $\mathrm{rad}^{eqv}$ measures the radius of the smallest ball in which such support is contained, an immediate consequence is the following

**Theorem 4.2.** *Let $\Gamma$ be a finitely generated amenable group. Then*

$$\rho_\Gamma \circ \mathrm{A}_\Gamma \geq \mathrm{Føl}_\Gamma .$$

*Proof.* Since $\mathrm{A}_G(n) = \mathrm{rad}(1, \frac{1}{n}) = \mathrm{rad}^{eqv}(1, \frac{1}{n})$, the number $\rho_\Gamma(\mathrm{A}_\Gamma(n))$ is the volume of the ball containing $\operatorname{supp} f$, where $f$ minimizes $\mathrm{rad}^{eqv}(1, \frac{1}{n})$. Thus

$$\rho_\Gamma(\mathrm{A}_\Gamma(n)) \geq \#\operatorname{supp} f \geq \mathrm{Føl}_\Gamma(n),$$

since $\mathrm{Føl}_\Gamma(n)$ minimizes the volume of $\operatorname{supp} f$ for $\varepsilon = \frac{1}{n}$. □

It follows that the function $\mathrm{A}_\Gamma$ in the case of amenable groups can have nontrivial behavior, as we now explain. Given two finitely generated groups $\Gamma_1$ and $\Gamma_2$ one defines their wreath product

$$\Gamma_1 \wr \Gamma_2 = \left( \oplus_{\gamma \in \Gamma_1} \Gamma_2 \right) \rtimes \Gamma_2,$$

where the action of $\Gamma_2$ on $(\oplus_{\gamma\in\Gamma_1}\Gamma_2)$ is by a coordinate shift. Since the wreath product preserves amenability, one can wonder how does the function $\mathrm{Føl}_{\Gamma_1\wr\Gamma_2}$ depend on the functions $\mathrm{Føl}_{\Gamma_1}$ and $\mathrm{Føl}_{\Gamma_2}$. This was studied in [Ver], [PS-C$_1$], [GŻ] and a complete answer was given by A. Erschler in [Er], where she proved that

$$\mathrm{Føl}_{\Gamma_1\wr\Gamma_2} \sim \left( \mathrm{Føl}_{\Gamma_1} \right)^{\mathrm{Føl}_{\Gamma_2}}, \tag{2}$$

provided that the following condition holds: ($\star$) *for any $C > 0$ there is a $K > 0$ such that for any $n > 0$,* $\mathrm{Føl}_{\Gamma_2}(Kn) > C\,\mathrm{Føl}_{\Gamma_2}(n)$. This last assumption will be automatically fulfilled in the cases we will consider, note however that it does not allow $\Gamma_2$ to be finite.

Now, using Theorem 4.2, we can relate this to the weak Følner function.

**Proposition 4.3.** *Let $\Gamma_1, \Gamma_2$ be discrete amenable groups and let* $\mathrm{Føl}_{\Gamma_2}$ *satisfy condition* ($\star$)*. Then*

$$\mathrm{A}_{\Gamma_1\wr\Gamma_2} \geq \mathrm{Føl}_{\Gamma_2}\left( \ln \mathrm{Føl}_{\Gamma_1} \right).$$

The proof amounts to recalling the fact that growth of a finitely generated group is at most exponential. Consequently, since for $G_k = \underbrace{\mathbb{Z} \wr (\ldots (\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z}) \ldots)}_{k \text{ times}}$ the Følner function satisfies

$$\mathrm{Føl}_{G_k} \sim \underbrace{n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}}_{k \text{ times}},$$

we obtain

**Corollary 4.4.** *Let $G_n$ be as above. Then*

$$\mathrm{A}_{G_k} \geq \underbrace{n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}}_{k-1 \text{ times}} \ln n.$$

Another example in [Er] is one of a group $\Gamma$ with $\mathrm{Føl}_\Gamma$ growing faster than any of the above iterated exponents. This of course gives the same conclusion for the function $\mathrm{A}_\Gamma$.

Recall also that it is not known whether Property A is satisfied for Thompson's group $F$. On the other hand it is known that the iterated wreath product

$$W_k = \underbrace{(\dots(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}) \wr \dots) \wr \mathbb{Z}}_{k \text{ times}}$$

is a quasi-isometrically embedded subgroup of $F$ for every $k \in \mathbb{N}$, this was shown by S. Cleary [Cl] (the article [Cl] covers just the case of $\mathbb{Z} \wr \mathbb{Z}$ and one needs to extend the argument presented in there in order to get the same statement for the iterated wreath products. I am very grateful to Sean Cleary for telling me about his results) and together with Proposition 3.7 and Theorem 3.11 leads to the following statement:

**Corollary 4.5.** *If Thompson's group $F$ has Property A then*

$$\mathrm{A}_F \geq n^k$$

*for every $k \in \mathbb{N}$.*

## 5. Asymptotic dimension and $\mathrm{A}_\Gamma$

In this section we will show another method to estimate $\mathrm{A}_X$, it is based on the connection between Property A and asymptotic dimension. In particular we'll show a large class of spaces for which $\mathrm{A}_X \sim n$. These spaces will arise as spaces with finite asymptotic dimension of *linear type*, i.e. where the diameter of the elements of the covers depends linearly on disjointness.

A family $\mathcal{U}$ of subsets of a metric space will be called $\delta$-bounded if $\operatorname{diam} U \leq \delta$ for every $U \in \mathcal{U}$. Two families $\mathcal{U}_1$, $\mathcal{U}_2$ are $R$-disjoint if $d(U_1, U_2) \geq R$ for any $U_1 \in \mathcal{U}_1$, $U_2 \in \mathcal{U}_2$.

**Definition 5.1** ([Gro$_2$])**.** *We say that a metric space $X$ has asymptotic dimension less than $k \in \mathbb{N}$, denoted* $\operatorname{asdim} X \leq k$*, if for every $R < \infty$ one can find a number $\delta < \infty$ and $k + 1$ $R$-disjoint families $\mathcal{U}_0, ..., \mathcal{U}_k$ of subsets of $X$ such that*

$$X = \mathcal{U}_0 \cup ... \cup \mathcal{U}_k$$

*and every $\mathcal{U}_i$ is $\delta$-bounded*

Asymptotic dimension is a large-scale version of the classical covering dimension in topology. It is a coarse invariant and a fundamental notion for [Yu$_1$], where the Novikov Conjecture for groups with finite asymptotic dimension is proved. Because of this result asymptotic dimension of groups has become a very actively studied notion, we refer the reader to the articles [BD$_1$], [BD$_2$] and to [Roe$_1$] and the references there for more on asymptotic dimension of finitely generated groups. Let us just mention here that examples of groups with finite asdim include free, hyperbolic, Coxeter groups, free products and extensions of groups with finite asdim. On the other hand it is easy to see that there are finitely generated groups which don't have finite asymptotic dimension - just take $\mathbb{Z} \wr \mathbb{Z}$ or Thompson's group $F$, each of which contains $\mathbb{Z}^k$ as a subgroup for every $k$ and since such inclusion is always a coarse embedding and $\operatorname{asdim} \mathbb{Z}^n = n$, it pushes asymptotic dimension off to infinity.

The following finer invariant associated to a space with finite asymptotic dimension was also introduced by Gromov [$\mathrm{Gro}_2$, p. 29], see also [$\mathrm{Roe}_1$, Chapter 9], [DZ, Section 4]

**Definition 5.2.** *Let $X$ be a metric space satisfying* $\operatorname{asdim} X \leq k$. *Define the type function $\tau_{k,X} : \mathbb{N} \to \mathbb{N}$ in the following way: $\tau_{k,X}(n)$ is the smallest $\delta \in \mathbb{N}$ for which $X$ can be covered by $k+1$ families $\mathcal{U}_0, ..., \mathcal{U}_k$ which are all $n$-disjoint and $\delta$-bounded.*

The type function is also known as *dimension function* and it's linearity is often referred to as *Higson property* or *finite Assouad-Nagata dimension*, see the discussion in Section 8. The proof of our next statement adapts an argument of Higson and Roe [HR], who showed that finite asymptotic dimension implies Property A.

**Theorem 5.3.** *Let $X$ be a metric space satisfying* $\operatorname{asdim} X \leq k$. *Then*

$$\mathrm{A}_X \preceq \tau_{k,X} \ .$$

*Proof.* By assumption, for every $n \in \mathbb{N}$, $X$ admits a cover by $k+1$, $\tau_{k,X}(n)$-bounded, $n$-disjoint families $\mathcal{U}_i$, as in definition 5.1. Let $\mathcal{U}$ be a cover of $X$ consisting of all the sets from all the families $\mathcal{U}_i$. There exists a partition of unity $\{\psi_V\}_{V\in\mathcal{U}}$ and a constant $C_k$ depending only on $k$ such that:

(i) each $\psi$ is Lipschitz with constant $2/n$;

(ii) $\sup \operatorname{diam}(\operatorname{supp} \psi) \leq \tau_{k,X}(n) + 4n \leq C_k \tau_{k,X}(n)$;

(iii) for every $x \in X$ no more than $k+1$ of the values $\psi(x)$ are non-zero.

For every $\psi$ choose a unique point $x_\psi$ in the set $\operatorname{supp}\psi$ and define

$$\xi_x^n = \sum_{\psi} \psi(x)\, \delta_{x_\psi}.$$

Then if $d(x,y) \leq 1$ we see that

$$\|\xi_x^n - \xi_y^n\|_{\ell_1(X)} = \sum_{\psi} |\psi(x) - \psi(y)| \ \leq \ \frac{2}{n} C_k',$$

where $C_k'$ is another constant depending on $k$ only and

$$\operatorname{supp}\ \xi_x^n \subseteq B(x, C_k \tau_{k,X}(C_k' n)).$$

Once again by Lemma 3.9 we're done. □

Thus spaces and groups of finite asymptotic dimension of linear type have $\mathrm{A}_X$ linear. The simplest examples of such are Euclidean spaces and trees, and their finite cartesian products, by an argument similar to the one in Proposition 3.8. It is also well-known that $\delta$-hyperbolic groups are in this class, one can quickly deduce this fact either directly from [$\mathrm{Roe}_2$] or from a theorem of Buyalo and Schroeder

[$BS_3$], which states that every hyperbolic group admits a quasi-isometric embedding into a product of a finite number of trees. In fact, Dranishnikov and Zarichnyi showed that every metric space with finite asymptotic dimension is equivalent to a subset of a product of a finite number of trees [DZ], however this equivalence is in general just coarse and not quasi-isometric, we will give examples illustrating this in Section 8.

## 6. The Main Theorem

As a corollary of the results presented in the two previous sections we get our main application, a direct relation between two of the considered large-scale invariants: Vershik's Følner function and Gromov's type of asymptotic dimension.

**Theorem 6.1.** *Let $\Gamma$ be a finitely generated amenable group satisfying* $\operatorname{asdim}\Gamma \leq k$. *Then there exists a constant $C$ depending only on $k$ such that*

$$\mathrm{Føl}_\Gamma \leq \rho_\Gamma \circ C\,\tau_{k,\Gamma}.$$

*Proof.* The estimate follows from Theorem 4.2 and Theorem 5.3. ◻

A general conclusion coming from this result is that several asymptotic invariants considered in the literature, namely: decay of the heat kernel, isoperimetric profiles, Følner functions, type function of asymptotic dimension, our function $\mathrm{A}_\Gamma$ and distortion of coarse embeddings, in the case of amenable groups all carry very similar information. We will show below how to use this fact to obtain results in various directions.

**Remark 6.2.** The constant $C$ in the above formula is a technical consequence of the estimates in the proof of Theorem 5.3 and it doesn't seem that we can get rid of it a priori. We can however omit it once we know for example that $\tau$ satisfies condition ($\star$) from Section 4: for every $C$ there exists a number $K$ such that $C\tau_{k,X}(n) \leq \tau_{k,X}(Kn)$ for all $n$. This is a very mild condition, in particular it holds for all common asymptotics. Another situation when the constant $C$ does not play a role is when the upper estimate on the growth $\rho_\Gamma$ is known. For the purposes of applications in Sections 7 and 8 we will be interested only in groups with exponential growth and we will omit the constant $C$ from now on.

## 7. Estimates of isoperimetric profiles

We will use our main theorem and asymptotic dimension to get precise estimates of the function $\mathrm{Føl}_\Gamma$ for some groups. Although these estimates are known (see e.g. [PS-$C_1$]), our purpose is to convince the reader that even though in Theorem 6.1 we, loosely speaking, pass between the volume of a set and the volume of the ball which contains it, which one can expect will cause some loss of information in the exponential growth case, we can in fact obtain sharp estimates on $\mathrm{Føl}_\Gamma$. To do this we will use the following consequence of Theorem 6.1.

**Corollary 7.1.** *If $\Gamma$ is an amenable group with exponential growth and finite asymptotic dimension of linear type then*

$$\mathrm{Føl}_\Gamma \sim e^n\,.$$

The statement follows from Theorem 6.1 and a theorem of Coulhon and Saloff-Coste [CS-C], stating that for groups of exponential growth the function Føl grows at least exponentially.

It should be also pointed out that the question of existence of amenable groups with exponential growth and at most exponential Følner function was first asked by Kaimanovich and Vershik in [KV].

**Example 7.2.** The first example we consider are groups $\mathfrak{S}_A = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$, where $A \in \mathrm{SL}_2(\mathbb{Z})$ satisfies $|\operatorname{trace}(A)| > 2$, usually one takes just

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The group $\mathfrak{S}_A$ has exponential growth and it is a discrete, quasi-isometrically embedded lattice in the group Sol used by Thurston to describe one of the geometries in his geometrization conjecture. The group Sol is quasi-isometric to a undistorted horosphere in $\mathbb{H}^2 \times \mathbb{H}^2$, a product of two hyperbolic planes. The latter has finite asymptotic dimension of linear type, this can be seen directly or from the fact that the hyperbolic plane embeds quasi-isometrically into a product of trees ([$\mathrm{BS}_2$]), and so we recover (see e.g. [PS-$\mathrm{C}_1$, Section 3]) the estimate

$$\text{Føl}_{\mathfrak{S}_A} \sim e^n.$$

**Example 7.3.** The solvable Baumslag-Solitar groups,

$$\mathrm{BS}(1,k) \;=\; \langle\, a, b \;:\; aba^{-1} = b^k \,\rangle,$$

where $k > 1$, constitute our second example. These groups are metabelian but not polycyclic and they act properly, cocompactly by isometries on a warped product $X_k = \mathbb{R} \times T_k$, where $T_k$ is an infinite, oriented, $k+1$-regular tree. For every vertex $v$ in this tree we have 1 incoming edge and $k$ edges going out of $v$, and we orient the incoming edge towards the vertex $v$. Metrically, the set $\mathbb{R} \times r$ where $r$ is an infinite, coherently oriented line, is an isometric copy of the hyperbolic plane, see [FM] for a detailed construction of the space $X_k$. Since both the tree and the hyperbolic plane have finite asymptotic dimension of linear type, it is easy to check by a direct construction of coverings or of a quasi-isometric embedding into an appropriately chosen space that $X_k$ also has finite asymptotic dimension of linear type. Thus, since by the Milnor-Švarc Lemma $\mathrm{BS}(1,k)$ is quasi-isometric to $X_k$, we get (see [PS-$\mathrm{C}_1$, Theorem 3.5])

$$\text{Føl}_{\mathrm{BS}(1,k)} \sim e^n.$$

**Example 7.4.** Assume we are given two finitely generated amenable groups $G$ and $H$ and an exact sequence

$$0 \longrightarrow K \longrightarrow \Gamma \longrightarrow H \longrightarrow 0, \tag{3}$$

i.e. $\Gamma$ is an extension of $K$ by $H$. Assume also that $K$ is undistorted in $\Gamma$ (recall that a subgroup $H$ is *undistorted* in the ambient group $\Gamma$ if the embedding of $H$ as a subgroup is quasi-isometric) and that both $K$ and $H$ have finite asymptotic dimension of linear type. Under these assumptions, in [BDLM] a Hurewicz-type theorem for asymptotic dimension of linear type is proved, which in particular implies that $\Gamma$ also has finite asymptotic dimension of linear type. In our situation this yields the following

**Corollary 7.5.** *Let $K$,$\Gamma$,$H$ be finitely generated amenable groups, sequence (3) be exact. Assume that $K$ is undistorted in $\Gamma$ and that the latter has exponential growth. If $H$ and $K$ have finite asymptotic dimension of linear type then*

$$\mathrm{Føl}_\Gamma \sim e^n.$$

Note however that this doesn't apply to the group $\mathfrak{S}_A$ considered above. In that example the fiber $\mathbb{Z}^2$ is well-known to be exponentially distorted in the ambient extension.

The above of course raises the question, which amenable groups with exponential growth have finite asymptotic dimension of linear type? Section 8 is devoted to building examples which badly fail this condition. One speculation however is the following. A group $\Gamma$ has *Prüfer rank $\kappa$* if $\kappa$ is the smallest integer such that every finitely generated subgroup of $\Gamma$ can be generated by at most $\kappa$ elements. For example, if $\Gamma$ is metabelian, has exponential growth and no torsion then finiteness of the Prüfer rank is equivalent to the fact that $\Gamma$ does not contain $\mathbb{Z} \wr \mathbb{Z}$ as a subgroup. In [PS-C$_2$, Sect.8, Q. 3] the authors ask whether solvable groups with finite Prüfer rank satisfy $\mathrm{Føl}_\Gamma \sim e^n$. In our context one might ask the following question, suggested by L. Saloff-Coste: *do solvable groups of finite Prüfer rank have finite asymptotic dimension of linear type?*

## 8. Applications to dimension theory

There are two folk questions concerning asymptotic dimension and its type function:

(Q.1) *How to build natural examples of finitely generated groups with $\tau_{k,\Gamma}$ growing faster than linearly for some $k$?* Most of the known examples of groups with finite asymptotic dimension have linear type and to the author's best knowledge no examples of groups with other behavior of the type function were known.

(Q.2) *Assume we have an example like in (Q.1), with* $\operatorname{asdim}\Gamma \leq k$ *and* $\tau_{k,\Gamma} > n$. *Can we find $k' > k$ such that $\tau_{k',\Gamma}$ will be linear?*

These questions, although quite natural, become even more relevant if one identifies after Dranishnikov and Zarichnyi [DZ, Section 4] asdim with linear type as the large-scale analog of the Assouad-Nagata dimension [As], [LS], which is an invariant in the Lipschitz category of metric spaces. The precise definition in our setting is simply the following: *a metric space $X$ has Assouad-Nagata dimension $\leq k$ if it satisfies* $\operatorname{asdim} X \leq k$ *and* $\tau_{k,X} \leq n$. The above questions can be then rephrased in the following way: (Q.1) *How to build finitely generated groups with Assouad-Nagata dimension strictly greater than asymptotic dimension?* (Q.2) *Does finite asymptotic dimension imply finite Assouad-Nagata dimension?*

We will use Theorem 6.1 to answer both questions and build some interesting examples of groups with finite asymptotic dimension. For any non-trivial finite group $H$ and for $k = 1, 2, 3, \ldots$ consider the group $\Gamma_k^{(1)} = H \wr \mathbb{Z}^k$. In the simplest case $H = \mathbb{Z}/2\mathbb{Z}$ the group $\Gamma_k$ is a lamplighter group (see e.g. [GŻ], [S-C$_1$]).

We have $\operatorname{asdim} \Gamma_k^{(1)} = k$. To see $\operatorname{asdim} \Gamma_k^{(1)} \leq k$ one needs to appeal to recent work of Dranishnikov and Smith [DS], in which they extend the notion of asymptotic dimension to all *countable* groups. And so observe that by [DS, Theorem 2.1], the infinitely generated countable group $\oplus_{z\in\mathbb{Z}^k} H$ (equipped with a proper length function inherited from $\Gamma_k^{(1)}$) has asymptotic dimension zero, since every of its finitely generated subgroups is finite. Since $\mathbb{Z}^k$ has asymptotic dimension $k$, the semi-direct product $(\oplus_{z\in\mathbb{Z}^k} H) \rtimes \mathbb{Z}^k$ is of asymptotic dimension at most $k$, by the Hurewicz-type theorem in [DS]. Then the inclusion of $\mathbb{Z}^k$ in $\Gamma_k^{(1)}$ as a subgroup gives $\operatorname{asdim} \Gamma_k^{(1)} \nleq k-1$.

Now, by Equation (2) in Section 4 we have

$$\text{Føl}_{\Gamma_k^{(1)}} \sim (\text{Føl}_H)^{\text{Føl}_{\mathbb{Z}^k}} \sim e^{(n^k)}.$$

For any $k' \geq k$, Theorem 6.1 gives

$$e^{(n^k)} \leq \rho_{\Gamma_k^{(1)}} \circ \tau_{k', \Gamma_k^{(1)}},$$

but this implies

$$n^k \leq \tau_{k', \Gamma_k^{(1)}},$$

since the growth of $\Gamma_k^{(1)}$ is exponential.

Now take the group $\Gamma_k^{(2)} = H \wr \Gamma_k^{(1)}$. By the same argument as before $\operatorname{asdim} \Gamma_k^{(2)} = k$, and again by Theorem 6.1 for any $k'$ we get

$$e^{\left(e^{(n^k)}\right)} \leq \rho_{\Gamma_k^{(2)}} \circ \tau_{k', \Gamma_k^{(2)}},$$

which gives

$$e^{(n^k)} \leq \tau_{k', \Gamma_k^{(2)}}.$$

Iterating this construction we get for a fixed $k$ and $i = 1, 2, \ldots$ infinitely many (depending on different choices of $H$) finitely generated groups $\Gamma_k^{(i)}$ with asdim equal exactly $k$ and type function growing at least as fast as the iterated exponential function

$$\underbrace{\exp\, \exp \ldots \exp}_{i-1 \text{ times}} n^k.$$

This gives the examples postulated by (Q.1) and answers (Q.2) negatively, since in particular all estimates are independent of $k'$.

Two comments are in order.

**Remark 8.1.** In the case of asymptotic dimension 1, the construction above is optimal in the following sense. Januszkiewicz and Świątkowski [JŚ] and independently Gentimis [Ge] proved that if a finitely presented group $G$ has asymptotic dimension 1 then it is virtually free, and it follows that it satisfies $\tau_{1,G} \leq n$. So the groups

$\Gamma_1^{(i)}$ for $i \geq 2$ are examples showing that results of Januszkiewicz-Świątkowski and Gentimis will not be true if one drops the requirement of finite presentation. It also follows that one cannot obtain examples with properties like $\Gamma_1^{(i)}$ and which would be finitely presented.

**Remark 8.2.** By [DZ] all the groups considered in this section embed coarsely into a product of finitely many trees. It might be interesting to note that by arguments similar to those in Theorem 3.11, any such embedding must be strongly distorting, i.e. for $\Gamma_k^{(i)}$ it must satisfy

$$\varphi_- \leq n^{\frac{1}{k}} \qquad \text{for } \; i = 1$$

and

$$\varphi_- \leq (\underbrace{\ln \ln \ldots \ln}_{i-1 \text{ times}} n)^{\frac{1}{k}} \qquad \text{for } \; i = 2, 3, \ldots$$

This contrasts again to the case of hyperbolic groups, which, as mentioned previously, embed quasi-isometrically into an appropriately chosen product of finitely many trees [$\mathrm{BS}_2$].

Thus, here's a question. Assume we have any subset $X \subset \mathcal{H}$ of an infinite-dimensional Hilbert space, satisfying $\operatorname{asdim} X \leq k < \infty$. Is it true that $\tau_{k,X} \leq n$? If so, conclusions similar to the ones in this remark would hold for embeddings into the Hilbert space. A similar statement is not true for $\mathrm{A}_\Gamma$, we will discuss the example below, in the section on compression.

## 9. Further applications and some questions

**$\mathrm{A}_\Gamma$ and Hilbert space compression.** As mentioned previously, Property A was introduced as a condition implying coarse embeddability. The *Hilbert space compression of $X$*, defined by Guentner and Kaminker [$\mathrm{GK}_2$], is the supremum of all the $0 \leq \alpha \leq 1$ such that the lower bound in Definition 2.2 satisfies $\varphi_- \geq n^\alpha$. See also [AGS], [CN], [$\mathrm{BS}_1$].

Observe that the function $\mathrm{A}_X$ gives estimates from below on the compression, since using the standard construction of a coarse embedding adapted to the space $\ell_1$ (see [$\mathrm{Yu}_2$], [$\mathrm{No}_1$]) we can construct a sequence of embeddings into $\ell_1$ with $\varphi_- \geq (\mathrm{A}_X^{-1})^\alpha$ for every $\alpha < 1$. Then, since $\ell_1$ has compression $\frac{1}{2}$ in $\ell_2$, by composing we get a family of coarse embedding into $\ell_2$ satisfying $\varphi_- \geq (\mathrm{A}_X^{-1})^{\alpha/2}$. For example, if $\mathrm{A}_X$ is linear this means that the compression number is at least $\frac{1}{2}$.

It is easy to see however that these estimates are not sharp - indeed, for hyperbolic groups we get an estimate on compression from below by $\frac{1}{2}$, but it was shown in [$\mathrm{BS}_1$] that hyperbolic groups have Hilbert space compression 1 (see also [$\mathrm{GK}_2$] for the same fact for the free group). However even though $\mathrm{A}_\Gamma$ does not give the best possible estimates, it can detect non-zero compression (cf. Question 1.12 in [AGS]), for example in the following sense: *if $\mathrm{A}_X$ is polynomial then the Hilbert space compression of $X$ is non-zero.*

Assume now that a metric space has Property A. It is natural to ask whether the best compression comes from a 'Property A-embedding'. More precisely, does there exist a finitely generated group $\Gamma$ with Property A and a coarse embedding of $\Gamma$ into the Hilbert space for which the distortion $\varphi_-$ grows faster than for any embedding constructed from Property A? It is clear to us that the answer is positive. The example we have in mind are the groups $W_k = (\ldots(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}) \wr \ldots) \wr \mathbb{Z}$ discussed in Section 4 . Indeed, as mentioned there, $W_k$ is a quasi-isometrically embedded subgroup of the Thompson group $F$, which by [AGS] has compression $\frac{1}{2}$. Thus the compression of $W_k$ is at least $\frac{1}{2}$ regardless of $k$, but $A_{W_k} \geq n^k$, so for $k$ large enough any embedding constructed from Property A should have significantly worse distortion than of exponent $\alpha = \frac{1}{2}$. We don't however have a rigorous proof establishing these claims, since formally it is not clear at all whether our construction of the embedding from Property A is the most efficient one.

Note that $W_k$ also provide the examples mentioned at the end of Remark 8.2. Indeed, since each $\mathrm{A}_{W_k}$ is embedded into the Hilbert space with compression $\frac{1}{2}$, then for $k \geq 3$ the coarse copy $\widetilde{W}_k \subseteq \mathcal{H}$ of $W_k$ satisfies $\mathrm{A}_{\widetilde{W}_k} \succ n$.

We also want to point out that another estimate on compression of spaces of polynomial growth, which uses Property A can be found in [Te].

**Type of asymptotic dimension.** Apart from the ones already mentioned in sections 7 and 8 we have several questions that also seem interesting:

(Q.4) Polycyclic groups have finite asymptotic dimension by [BD$_2$]. For polycyclic groups of exponential growth we also have $\mathrm{Føl}_\Gamma \sim e^n$ (see [PS-C$_1$, Theorem 3.4]), so $\mathrm{Føl}_\Gamma$ does not obstruct linearity of the type of asymptotic dimension. Do polycyclic groups have finite asymptotic dimension of linear type?

(Q.5) Can one find examples similar to the ones in Section 8 but with subexponential volume growth? Note that groups of intermediate growth often have infinite asymptotic dimension.

(Q.6) What is the optimal upper bound on the type of $\Gamma_k^{(i)}$? In particular, is it true that $\tau_{k,H\wr\mathbb{Z}^k} \sim n^k$ for a finite group $H$?

(Q.7) Find examples of groups with finite asdim and strictly larger but still finite Assouad-Nagata dimension. If these exists, how large can the difference of these dimensions be?

**Random walks on non-amenable groups?** As mentioned already several times in the text, the Følner function of an amenable group can be connected with random walks on the group, it gives estimates on the decay of the heat kernel. In the case of non-amenable groups however we don't expect any connection between isoperimetry and weak Følner function or type of asymptotic dimension, since for example for hyperbolic groups the isoperimetric profile satisfies $J \sim \mathrm{const}$ while the other two functions are always linear.

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Department of Mathematics, Vanderbilt University, Nashville, TN, USA.
*E-mail address*: `piotr.nowak@vanderbilt.edu`