COARSE EMBEDDINGS OF METRIC SPACES INTO BANACH SPACES

PIOTR W. NOWAK

ABSTRACT. There are several characterizations of coarse embeddability of locally finite metric spaces into a Hilbert space. In this note we give such characterizations for general metric spaces. By applying these results to the spaces $L_p(\mu)$, we get their coarse embeddability into a Hilbert space for $0 < p < 2$. This together with a theorem by Banach and Mazur yields that coarse embeddability into ℓ_2 and into $L_p(0,1)$ are equivalent when $1 \leq p < 2$. A theorem by G.Yu and the above allow to extend to $L_p(\mu)$, $0 < p \leq 2$, the range of spaces, coarse embedding into which guarantees for a finitely generated group Γ to satisfy the Novikov Conjecture.

1. INTRODUCTION

Coarse embeddings were introduced by M. Gromov.

Definition 1.1 ([Gr, 7.E.]). Let X, Y be metric spaces. A (not necessarily continuous function) $f \colon X \to Y$ is a *coarse embedding* if there exist non-decreasing functions $\rho_1, \rho_2 \colon [0, \infty) \to [0, \infty)$ satisfying

(1) $\rho_1(d_X(x,y)) \leq d_Y(f(x), f(y)) \leq \rho_2(d_X(x,y))$ for all $x, y \in X$,
(2) $\lim_{t \to \infty} \rho_1(t) = +\infty$.

In the language of coarse geometry, a coarse embedding $f \colon X \to Y$ is a coarse equivalence of X and $f(X)$ (for the notions of coarse geometry we refer the reader to [Gr, Roe]).

Coarse embeddings have been in the spotlight recently due to a remarkable theorem by G. Yu [Yu], who showed that every discrete metric space Γ which embeds coarsely into a Hilbert space satisfies the Coarse Baum-Connes Conjecture. In particular, if Γ is a finitely generated group with word length metric coarsely embeddable into a Hilbert space and the classifying space $B\Gamma$ has a homotopy type of a finite CW-complex then the Novikov Conjecture holds for Γ.

Gromov asked [Gr, 7.E$_2$.] if every separable metric space embeds coarsely into a Hilbert space. A negative answer to this question was given by A.N. Dranishnikov, G. Gong, V. Lafforgue and G. Yu [DGLY]. The counterexample was constructed using ideas of P. Enflo, who answered negatively a question raised by J. Smirnov whether every separable metric space is uniformly homeomorphic to a subset of a Hilbert space [En].

Inspired by the work of Dranishnikov, Gong, Lafforgue and Yu [DGLY], we go somewhat further in this direction and try to find similarities between coarse embeddings and uniform embeddings (i.e., uniform homeomorphisms onto a subset).

As a result, we generalize characterizations of coarse embeddability into ℓ_2 given in [DGLY] in terms of negative definite functions from locally finite to general metric spaces. In particular we show, that coarse embeddability of an arbitrary metric space depends only on its finite subset. As an application we prove that for $0 < p < 2$ the space L_p admits a coarse embedding into a Hilbert space. This and the fact that the space $L_p(0,1)$ for $p > 1$ contains ℓ_2 as a subspace yields that coarse embeddability into ℓ_2 and into $L_p(0,1)$ are equivalent. All of the above combined with Yu's theorem on the Baum-Connes Conjecture [Yu] and the Descent Principle [Roe] implies obviously that the Novikov Conjecture holds for finitely generated groups coarsely embeddable into a separable $L_p(\mu)$-space, $0 < p < 2$.

Throughout this note we will talk about *coarse embeddability*, having in mind that the embedding is into a Hilbert space. We will also assume, that all Hilbert spaces are real - this makes no loss to generality, since embedding into a complex Hilbert space is equivalent to embedding into a real Hilbert space.

Let us also note that if $f \colon X \to \mathcal{H}$ is a coarse embedding then a reasonable assumption on X, allowing $\mathcal{H} = \ell_2$, is for X to contain an at most countable c-net $\mathcal{C}$ for some constant $c > 0$ (i.e., for every $x \in X$ there exists $y \in \mathcal{C}$ such that $d(x,y) \le c$). Of course, in the coarse category $\mathcal{C}$ and X are isomorphic. In particular, this is the case, when our metric space X is a (finitely generated) group Γ with a word length metric or (more generally) when X is separable.

2. Preliminaries

Positive definite functions appeared in the work of E.H. Moore [Mo]. Let us briefly recall basic definitions and facts concerning positive and negative definite kernels and functions. For more details we refer the reader to [Sch₂], [BL, Chapter 8] and [AMM].

Definition 2.1. By a kernel on a set X we mean any symmetric function $K : X \times X \to \mathbb{R}$. A kernel K is said to be:

(1) *positive definite* if $\sum K(x_i, x_j)c_i c_j \ge 0$ for all $n \in \mathbb{N}$ and $x_1, ..., x_n \in X$, $c_1, ..., c_n \in \mathbb{R}$.
(2) *negative definite* if $\sum K(x_i, x_j)c_i c_j \le 0$ for all $n \in \mathbb{N}$ and $x_1, ..., x_n \in X$, $c_1, ..., c_n \in \mathbb{R}$ such that $\sum c_i = 0$.

A function $f \colon X \to \mathbb{R}$ on a commutative metric group X is said to be positive (negative) definite if $K(x,y) = f(x-y)$ is a positive (negative) definite kernel.

A positive (negative) definite kernel K is said to be *normalized* if $K(x,x) = 1$ ($K(x,x) = 0$) for all $x \in X$. The set of positive definite kernels is closed under

limits in the function space $\mathbb{R}^{X \times X}$ with pointwise convergence.

The following theorem describes the relation between negative and positive definite kernels.

Theorem 2.2 (I.J. Schoenberg [Sch₂]; see also [BL, Proposition 8.4]). *A kernel N on X is negative definite if and only if e^{-tN} is positive definite for every $t > 0$.*

The key to study the connection between maps into Hilbert spaces and positive and negative definite kernels is the following theorem (see [BL, Proposition 8.5]).

Theorem 2.3 ((1) E.H. Moore [Mo]; (2) I.J. Schoenberg [Sch₂]). *(1) A kernel K on a set X is positive definite if and only if there exist a Hilbert space $\mathcal{H}$ and a map $f : X \to \mathcal{H}$ such that*

$$K(x, y) = \langle f(x), f(y) \rangle \quad \text{for all } x, y \in X.$$

(2) A normalized kernel N on X is negative definite if and only if there exist a Hilbert space $\mathcal{H}$ and a map $f : X \to \mathcal{H}$ such that

$$N(x, y) = \|f(x) - f(y)\|^2 \quad \text{for all } x, y \in X.$$

Part (1) of Theorem 2.3 is the main tool in the study of uniform embeddings (i.e., uniform homeomorphisms onto a subset) into Hilbert spaces (see [AMM, BL]) while part (2) gives, as an immediate consequence, the well-known Schoenberg's characterization of metric spaces which embed isometrically into a Hilbert space: *a metric space (X, d) embeds isometrically into a Hilbert space if and only if d^2 is a negative definite kernel on X.*

3. Coarse embeddings into Hilbert spaces and negative definite kernels

In this section we characterize coarse embeddability of metric spaces. In [DGLY] the authors give a condition equivalent to coarse embeddability of a locally finite metric space into a Hilbert space in terms of negative definite kernels [DGLY, Theorem 2.2] (recall that a metric space is locally finite if every ball has finitely many elements). This characterization remains true for general metric spaces.

Proposition 3.1. *A metric space X admits a coarse embedding into a Hilbert space if and only if there exist a normalized, negative definite kernel on X and non-decreasing functions $\rho_i : [0, \infty) \to [0, \infty)$, $i = 1, 2$, satisfying*

    (1) $\rho_1(d(x, y)) \le N(x, y) \le \rho_2(d(x, y))$;
    (2) $\lim_{t \to \infty} \rho_1(t) = \infty$.

Proof. Suppose, that there exists a kernel N satisfying conditions (1) and (2). By Theorem 2.3, there exist a Hilbert space $\mathcal{H}$ and a map $f : X \to \mathcal{H}$ such that

$$N(x, y) = \|f(x) - f(y)\|^2$$

for all $x, y \in X$. In other words, f satisfies

$$\sqrt{\rho_1(d(x, y))} \le \|f(x) - f(y)\| \le \sqrt{\rho_2(d(x, y))}.$$

Obviously, f is a coarse embedding.

Now suppose, that $f \colon X \to \mathcal{H}$ is a coarse embedding and define $N(x,y) = \|f(x) - f(y)\|^2$. Then, N is a normalized, negative definite kernel on X: when $\sum c_i = 0$ and $x_1, ..., x_n \in X$ then

$$\sum_{i,j=1}^{n} \|f(x_i)\|^2 c_i c_j \quad = \quad \left(\sum_{i=1}^{n} c_i \right) \left(\sum_{j=1}^{n} \|f(x_i)\|^2 c_j \right),$$

hence

$$\sum_{i,j=1}^{n} N(x_i, x_j) c_i c_j =$$

$$\sum_{i,j=1}^{n} \left(\|f(x_i)\|^2 + \|f(x_j)\|^2 - 2\langle f(x_i), f(x_j) \rangle \right) c_i c_j =$$

$$= -2 \left\| \sum_{i=1}^{n} c_i f(x_i) \right\|^2 \leq 0$$

$\square$

Remark 3.2. The proof of Theorem 2.3 and Theorem 2.2 [DGLY] use the same reasoning. In fact, the proof of the latter works also in case of general metric spaces, except that for not locally finite spaces the construction may result in a non-separable Hilbert space.

Proposition 3.1 leads to the following

Corollary 3.3. *Let (X, d) be a metric space. If for some non-decreasing function $\alpha \colon [0, \infty) \to [0, \infty)$ satisfying $\lim_{t \to \infty} \alpha(t) = \infty$ the kernel $\alpha \circ d$ is negative definite, then X admits a coarse embedding into a Hilbert space.*

Proof. If $\alpha \circ d$ is a negative kernel on X, then it satisfies the conditions of Proposition 3.1 with $\rho_1(t) = \rho_2(t) = \alpha(t)$. $\square$

Also in [DGLY] the authors prove that coarse embeddability in the case of locally finite metric spaces can be reduced to the question, whether all finite subsets can be mapped into a Hilbert space by functions having uniform estimates on the distance of images of two points [DGLY, Definition 3.1, Proposition 3.2].

We will show that again the local finiteness assumption can be dropped and that coarse embeddability of a metric space into a Hilbert space is determined by its finite subsets.

Theorem 3.4. *A metric space X admits a coarse embedding into a Hilbert space if and only if there exist non-decreasing functions $\rho_i \colon [0, \infty) \to [0, \infty)$, $i = 1, 2$, such that $\lim_{t \to \infty} \rho_1(t) = \infty$ and for every finite subset $A \subset X$ there exists a map $f_A \colon A \to \ell_2$ satisfying*

$$\rho_1(d(x,y)) \leq \|f_A(x) - f_A(y)\| \leq \rho_2(d(x,y))$$

for every $x, y \in X$.

Proof. Suppose that there exist functions ρ_1, ρ_2 and $f_A \colon A \to \ell_2$ as above. The kernel $N_A(x,y) = \|f_A(x) - f_A(y)\|^2$ is negative definite on A and we have

$$\rho_1^2(d(x,y)) \leq N_A(x,y) \leq \rho_2^2(d(x,y))$$

for all $x, y \in A$. Define a new kernel $K_A = e^{-N_A}$. By Theorem 2.2, the kernel K_A is positive definite and

$$e^{-\rho_2^2(d(x,y))} \leq K_A(x,y) \leq e^{-\rho_1^2(d(x,y))}.$$

Extend the kernel K_A to a positive definite kernel $\widetilde{K}_A$ on X, by defining it to be 0 outside the set $A \times A$. For all $x, y \in X$ we have $|\widetilde{K}_A(x,y)| \leq 1$, thus the closure of the set $\left\{ \widetilde{K}_A \colon \#A < \infty \right\}$ in the space $\mathbb{R}^{X \times X}$ with pointwise convergence topology is compact, by Tichonov's theorem. Let K be a pointwise limit of a convergent subnet of the net[1] $\left\{ \widetilde{K}_A \colon \#A < \infty \right\}$. Then K is a positive definite kernel on X, satisfying

$$e^{-\rho_2^2(d(x,y))} \leq K(x,y) \leq e^{-\rho_1^2(d(x,y))}$$

for all $x, y \in X$.

Since $K(x, y) > 0$ for all $x, y \in X$, we can set $N(x, y) = -\ln K(x, y)$. We will show that N is negative definite. Indeed, since K is a pointwise limit and the function e^{-x} is continuous, for every $t > 0$ we have

$$(1) \qquad e^{-tN(x,y)} = K(x,y)^t = \left(\lim \widetilde{K}_A(x,y) \right)^t = \lim \left(\widetilde{K}_A(x,y) \right)^t$$

where the limit is the one of the convergent subnet. By Theorem 2.2 the kernels $\widetilde{K}_A^t$ are positive definite for every $t > 0$ and the right-hand side of equation (1) is a positive definite kernel as a pointwise limit of positive definite kernels . Thus $e^{-tN(x,y)}$ is positive definite for every $t > 0$ and Theorem 2.2 yields that N is a negative definite kernel on X satisfying

$$\rho_1^2(d(x,y)) \leq N(x,y) \leq \rho_2^2(d(x,y)).$$

By Proposition 3.1, X admits a coarse embedding into a Hilbert space. The other implication is obvious. $\qquad\square$

4. COARSE EMBEDDINGS OF L_p-SPACES

In this section, as an application of the above results, we show that the space $L_p(\mu)$ is coarsely embeddable for $0 < p \leq 2$, which gives, as an immediate consequence, a stronger version of [DGLY, Proposition 3.3], where the authors showed, that every locally finite subspace of $\ell_1(\mathbb{N})$ admits a coarse embedding into a Hilbert space.

The spaces $L_p(\mu)$ are metric only for $p \geq 1$ but we will consider also the case $0 < p < 1$.

Proposition 4.1. *For any measure μ, the space $L_p(\mu)$ (in particular ℓ_p) admits a coarse embedding into a Hilbert space for $0 < p < 2$.*

To prove Proposition 4.1 we need the following lemma, which was first proved by I.J. Schoenberg [Sch$_1$, Theorem 2]

Lemma 4.2. *Let N be a negative definite kernel on X and $N(x, y) \geq 0$ for all $x, y \in X$. Then the kernel N^α is negative definite for any $0 < \alpha < 1$.*

---

[1]Here, by a *net* we mean a generalized sequence

Proof of the Lemma. Let N be a negative definite kernel. Then for every $t \geq 0$ the kernel $1 - e^{-tN} \geq 0$ is also negative definite, and we have

$$\int\limits_0^\infty \left(1 - e^{-tN}\right) d\mu(t) \geq 0$$

for every positive measure μ on $[0, \infty)$. For every $x > 0$ and $0 < \alpha < 1$ the following formula holds

$$x^\alpha = c_\alpha \int\limits_0^\infty \left(1 - e^{-tx}\right) t^{-\alpha-1} dt,$$

where c_α is some positive constant. Thus N^α is also a negative definite kernel for every $0 < \alpha < 1$. □

Proof of Proposition 4.1. The kernel $|x - y|^2$ is negative definite on the real line (as a square of the metric on a Hilbert space). By the above Lemma, for any $0 < p \leq 2$ the kernel $|x - y|^p$ also is negative definite on $\mathbb{R}$, i.e.,

$$\sum |x_i - x_j|^p c_i c_j \leq 0$$

for every such p, all $x_1, ..., x_n \in \mathbb{R}$ and $c_1, ..., c_n \in \mathbb{R}$ such that $\sum c_i = 0$. By integrating the above inequality with respect to measure μ, we get that the function $\|x\|^p$ is negative definite on $L_p(\mu)$ and the assertion follows from Corollary 3.3. □

Corollary 4.3. *If a metric space X admits a coarse embedding into the space $L_p(\mu)$ for $0 < p < 2$ and some measure μ (in particular into ℓ_p), then X admits a coarse embedding into a Hilbert space.*

G.Yu showed that every discrete metric space Γ with bounded geometry which admits a coarse embedding into a Hilbert space satisfies the Coarse Baum-Connes Conjecture [Yu]. In particular, if Γ is a finitely generated group with word length metric then the Descent Principle [Roe] implies that the Novikov Conjecture holds for Γ. To situate the results of this section in the context of the Novikov Conjecture we give this obvious

Corollary 4.4. *Let Γ be a finitely generated group and let the classifying space $B\Gamma$ have the homotopy type of a finite CW-complex. If Γ (as a metric space with word length metric) admits a coarse embedding into a (separable) space $L_p(\mu)$, $0 < p \leq 2$ then the Novikov Conjecture holds for Γ.*

Let us also note an interesting consequence of a theorem due to S.Banach and S.Mazur on linear dimension of the spaces L_p [Ba, Theorem 6, p.123] which states that ℓ_2 is isomorphic to a subspace $L_p(0, 1)$ if $p > 1$. From this we conclude, that every separable metric space, coarsely embeddable into a Hilbert space, admits a coarse embedding into any $L_p(0, 1)$ with $p \geq 1$. Moreover, for $1 < p \leq 2$ the space $L_p(0, 1)$ is isometric to a subspace of $L_1(0, 1)$ ([LT, Theorem II.3.14, p.139]). Thus, we arrive at the following

Corollary 4.5. *Let X be a separable metric space. The following statements are equivalent:*

(1) *X is coarsely embeddable into ℓ_2;*
(2) *X is coarsely embeddable into $L_p(0, 1)$ for some (equivalently all) $1 \leq p < 2$.*

5. Final remarks

(A) Let $\mathcal{C}$ denote the set of all integer sequences which are eventually zero. This is the abelian group $\oplus_{n \in \mathbb{N}} \mathbb{Z}$ on which the word length metric (with respect to the standard generators) is just the ℓ_1 metric. With this metric $\mathcal{C}$ is coarsely embeddable into ℓ_2 by Proposition 4.1. There is another natural metric on $\mathcal{C}$, induced by the sup-norm $\| \cdot \|_0$. We claim that $(\mathcal{C}, \| \cdot \|_0)$ is not coarsely embeddable into ℓ_2.

To show this consider $(\mathcal{C}, \| \cdot \|_0)$ as a 1-net in the space c_0. If $(\mathcal{C}, \| \cdot \|_0)$ would be coarsely embeddable into ℓ_2 then so would be c_0. By a result of I. Aharoni [Ah] there exists a constant $K > 0$ such that for every separable metric space X there is a map $T \colon X \to c_0$ satisfying

$$d(x, y) \leq \|Tx - Ty\| \leq Kd(x, y) \quad \text{for all } x, y \in X.$$

Therefore any separable metric space would be coarsely embeddable into ℓ_2, which is not the case by a result in [DGLY]. This contradiction completes the proof.

Let us note that, unfortunately, the group $\mathcal{C}$ is infinitely generated and fails to be locally finite.

(B) The results in Section 4 are counterparts of facts from the theory of "classical" uniform embeddings into Hilbert space (see [AMM, BL]). A theorem of I. Aharoni, B. Maurey and B.S. Mityagin [AMM, Corollary 4.3] states, that $L_p(\mu)$ is not uniformly homeomorphic to any subset of a Hilbert space if $p > 2$. It is natural to ask, whether this result has a counterpart for coarse uniform embeddings, i.e., does the space $L_p(\mu)$ admit a coarse uniform embedding into a Hilbert space if $p > 2$?

(C) Proposition 3.1 can provide an alternative proof of the characterization of coarse embeddability, given by M. Dadarlat and E. Guentner [DG, Proposition 2.1]: *a metric space X is coarsely embeddable into a Hilbert space $\mathcal{H}$ if and only if for every $R > 0$ and every $\varepsilon > 0$ there exist $S > 0$ and a map $f \colon X \to \mathcal{H}$ such, that for all $x, y \in X$ we have $\|f(x)\| = 1$, $\sup\{ \|f(x) - f(y)\| \colon d(x, y) \leq R \} \leq \varepsilon$ and $\lim_{S \to \infty} \inf\{ \|f(x) - f(y)\| \colon d(x, y) \geq S \} = \sqrt{2}$.* We leave the details to the reader.

(D) Since $c_0^* = \ell_1$ and $\ell_1^* = \ell_\infty \supset c_0$, Proposition 4.1 and the above discussion provide an example of a Banach space, which is coarsely embeddable, while it's dual is not, and of a Banach space, which is not coarsely embeddable, while it's dual is.

References

[Ah]   I. Aharoni, *Every separable metric space is Lipschitz equivalent to a subset of c_0*, Isr. J. Math. 19 (1974) 284-291.

[AMM]   I. Aharoni, B. Maurey, B. S.Mityagin, *Uniform embeddings of metric spaces and of Banach spaces into Hilbert Spaces*, Isr. J. Math., 52 (1985), 251-265.

[Ba]   S. Banach, *Theory of Linear Operations*, North-Holland Mathematical Library, Volume 38 (1987).

[BL]   Y. Benyamini, J. Lindenstrauss, *Geometric nonlinear functional anlysis*, Volume 48 of Colloquium Publications. American Mathematical Society, Providence, R.I., 2000.

[DG]   M. Dadarlat, E. Guentner, *Constructions preserving Hilbert space uniform embeddability of discrete groups*, Preprint, 2002.

[DGLY]   A.N. Dranishnikov, G. Gong, V. Lafforgue, G. Yu, *Uniform Embeddings into Hilbert Space and a Question of Gromov*, Canad. Math. Bull., Vol.45(1), 2002, 60-70.

[En]   P. Enflo, *On a problem of Smirnov*, Ark. Math. 8 (1969), 107-109.

[Gr]   M. Gromov, *Asymptotic invariants of infinite groups*, London Mathematical Society Lecture Notes, no.182, s. 1-295, Cambridge University Press, 1993.

[LT]   J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces*, Springer-Verlag Lecture Notes in Mathematics 338 (1973).

[Mo]   E.H. Moore, *On properly positive Hermitian matrices*, Bull.Am. Math. Soc. 23 (1916), 59, 66-67.

[Roe]   J. Roe, *Index theory, coarse geometry, and topology of manifolds*, CBMS Regional Conference Series in Mathematics, AMS 1996.

[Sch$_1$] I.J. Schoenberg, *On certain metric spaces arising from euclidean spaces by a change of metric and their imbedding in Hilbert space*, Ann. Math. 38 (1937), 787-793.

[Sch$_2$] I.J. Schoenberg, *Metric spaces and positive definite functions*,Trans. Am. Math. Soc. 44 (1938), 522-536.

[Yu]   G. Yu, *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Invent. Math. (1) 139 (2000), 201-240.

Institute of Mathematics, Warsaw University, ul. Banacha 2, 02-097 Warsaw, Poland.

Department of Mathematics, Tulane University, 6823 St. Charles Avenue, New Orleans, LA 70118, USA.

Current address: Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, Nashville, TN 37240 USA.

E-mail address: `pnowak@math.vanderbilt.edu`