

ON COARSE EMBEDDABILITY INTO ℓ_p -SPACES AND A CONJECTURE OF DRANISHNIKOV

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ABSTRACT. We show that the Hilbert space is coarsely embeddable into any ℓ_p for $1 \leq p \leq \infty$. It follows that coarse embeddability into ℓ_2 and into ℓ_p are equivalent for $1 \leq p < 2$.

Coarse embeddings were defined by M. Gromov [Gr, 7.E₂] to express the idea of inclusion in the large scale geometry of groups. G. Yu showed later that the case when a finitely generated group with a word length metric is being embedded into the Hilbert space is of great importance in solving the Novikov Conjecture [Yu], while recent work of G. Kasparov and G. Yu [KY] treats the case when the Hilbert space is replaced with just a uniformly convex Banach space. Due to these remarkable theorems coarse embeddings gain a great deal of attention, but still embeddability into the Hilbert, and more generally Banach spaces, is not entirely understood with many question remaining open.

In this context the class of ℓ_p -spaces seems to be particularly interesting. Their embeddability into the Hilbert space is known - ℓ_p admits such an embedding when $0 < p \leq 2$ but do not if $p > 2$ due to a recent result of W. Johnson and N. Randrianarivony [JR]. In this note we study the opposite situation, i.e. we show that the separable Hilbert space embeds into ℓ_p for any $1 \leq p \leq \infty$. As a consequence we obtain a new characterization of embeddability into ℓ_2 , namely that the properties of embeddability into ℓ_p for $1 \leq p \leq 2$ are all equivalent.

In [GK, Section 6] the authors advertised a conjecture stated by A.N. Dranishnikov [Dr, Conjecture 4.4]: *a discrete metric space has Property A if and only if it admits a coarse embedding into the the space ℓ_1* . The results presented in this note show, that this is the same as asking whether Property A is equivalent to embeddability into the Hilbert space, and although it is a folk conjecture that such statement is not true, no example distinguishing between the two is known.

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L_p -SPACES AND THE MAZUR MAP

In everything what follows we consider only separable $L_p(\mu)$ -spaces and we will specialize to the most interesting case of the spaces ℓ_p , the case for $L_p(\mu)$ for other, including non-separable, measures follows easily and is left to the reader. We use the standard notation $\ell_p = \ell_p(\mathbb{N})$ and by $S(X)$ we denote the unit sphere in the Banach space X .

The Mazur map $M_{p,q} : S(\ell_p) \rightarrow S(\ell_q)$ is defined by the formula

$$M_{p,q}(x) = \left\{ |x_i|^{\frac{p}{q}} \operatorname{sign} x_i \right\}_{i=1}^{\infty}$$

where $x = \{x_i\}_{i=1}^{\infty} \in \ell_p$. It is a uniform homeomorphism between unit spheres of ℓ_p -spaces. More precisely, for some C depending only on $\frac{p}{q}$ it satisfies the inequalities:

$$(1) \quad \frac{p}{q} \|x - y\|_p \leq \|M_{p,q}(x) - M_{p,q}(y)\|_q \leq C \|x - y\|_p^{p/q}$$

for all $x, y \in S(\ell_p)$ and $p < q$, and the opposite inequalities if $p > q$ (note that $M_{p,q} = M_{q,p}^{-1}$). For the proof of these estimates and details on the Mazur map and its applications we refer the reader to [BL, Chapter 9.1].

If $\{X_n\}_{n \in \mathbb{N}}$ is a sequence of Banach spaces, we denote by $(\sum X_n)_p$ the direct sum of X_n with the p -norm, i.e.

$$\left(\sum_{n=1}^{\infty} X_n \right)_p = \left\{ \mathbf{x} = \{x_n\}_{n \in \mathbb{N}} : x_n \in X_n, \sum_{n=1}^{\infty} \|x_n\|^p < \infty \right\},$$

$$\|\mathbf{x}\|_p = \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{\frac{1}{p}}.$$

Clearly, ℓ_p is isometric to $(\sum \ell_p)_p$

We will also need the following classification of separable L_p -spaces.

Theorem 1 (see e.g. [Wo, III.A]). *A separable space $L_p(\mu)$ is isometric to one of the following spaces: ℓ_p^n for $n = 1, 2, 3, \dots$, $L_p[0, 1]$, ℓ_p , $(L_p[0, 1] \oplus \ell_p^n)_p$ for $n = 1, 2, 3, \dots$, $(L_p[0, 1] \oplus \ell_p)_p$.*

A CONDITION FOR COARSE EMBEDDABILITY

We recall the definition of a coarse embedding.

Definition 1. Let X, Y be metric spaces. A map $f: X \rightarrow Y$ is a *coarse embedding* if there exist non-decreasing functions $\rho_1, \rho_2: [0, \infty) \rightarrow [0, \infty)$ satisfying

- (1) $\rho_1(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_2(d_X(x, y))$ for all $x, y \in X$,
- (2) $\lim_{t \rightarrow \infty} \rho_1(t) = +\infty$.

In [DG] M. Dadarlat and E. Guentner characterized spaces coarsely embeddable into the Hilbert \mathcal{H} space in terms of existence of maps into the unit sphere $S(\mathcal{H})$. Their result is a reminiscence of a characterization of *uniform embeddability* (meaning an existence of a uniform homeomorphism onto a subset) into a Hilbert space obtained by Aharoni et al in [AMM].

Theorem 2 ([DG, Theorem 2.1]). *A metric space X admits a coarse embedding into the Hilbert space \mathcal{H} if and only if for every $R > 0$ and $\varepsilon > 0$ there is a map $\varphi: X \rightarrow S(\mathcal{H})$ and $S > 0$ satisfying*

- (1) $\sup \{\|\varphi(x) - \varphi(y)\|_{\mathcal{H}} : x, y \in X, d(x, y) \leq R\} \leq \varepsilon$,
- (2) $\lim_{S \rightarrow \infty} \inf \{\|\varphi(x) - \varphi(y)\|_{\mathcal{H}} : x, y \in X, d(x, y) \geq S\} = \sqrt{2}$

We are going to use this idea to prove a similar condition for embeddings into the spaces ℓ_p . The proof relies on the original proof of Theorem 2.

Theorem 3. *Let X be a metric space and $1 \leq p < \infty$. If there is a $\delta > 0$ such that for every $R > 0, \varepsilon > 0$ there is a map $\varphi: X \rightarrow S(\ell_p)$ satisfying*

- (1) $\sup \{\|\varphi(x) - \varphi(y)\|_p : x, y \in X, d(x, y) \leq R\} \leq \varepsilon$,
- (2) $\lim_{S \rightarrow \infty} \inf \{\|\varphi(x) - \varphi(y)\|_p : x, y \in X, d(x, y) \geq S\} \geq \delta$,

then X admits a coarse embedding into ℓ_p .

Proof. By the assumptions for every $n \in \mathbb{N}$ there is a map $\varphi_n: X \rightarrow S(\ell_p)$ and a number $S_n > 0$ such that $\|\varphi_n(x) - \varphi_n(y)\|_p \leq \frac{1}{2^n}$ whenever $d(x, y) \leq n$ and $\|\varphi_n(x) - \varphi_n(y)\|_p \geq \frac{\delta}{2}$ whenever $d(x, y) \geq S_n$. Without loss of generality we can choose the sequence of S_n 's to be strictly increasing and tending to infinity as $n \rightarrow \infty$.

Choose $x_0 \in X$ and define a map $\Phi: X \rightarrow (\sum \ell_p)_p$ by the formula

$$\Phi(x) = \bigoplus_{n=1}^{\infty} (\varphi_n(x) - \varphi_n(x_0)).$$

It is easy to see that

$$\|\Phi(x)\|_p^p = \sum_{n=1}^{\infty} \|\varphi_n(x) - \varphi_n(x_0)\|_p^p < \infty$$

which shows that Φ is well-defined.

We will show that Φ is a coarse embedding. Take $k \in \mathbb{N}$ and $\sqrt[p]{k-1} \leq d(x, y) < \sqrt[p]{k}$. Then

$$\begin{aligned} \|\Phi(x) - \Phi(y)\|_p^p &= \sum_{n=1}^{k-1} \|\varphi_n(x) - \varphi_n(y)\|_p^p + \sum_{n=k}^{\infty} \|\varphi_n(x) - \varphi_n(y)\|_p^p \\ &\leq 2^p(k-1) + \sum_{n=k}^{\infty} \frac{1}{2^{kp}} \leq 2^p(k-1) + 1 \leq 2^p d(x, y)^p + 1 \end{aligned}$$

The first estimate comes from the fact that unit vectors cannot be more than distance 2 apart.

On the other hand for $S_{k-1} \leq d(x, y) < S_k$ we have

$$\|\Phi(x) - \Phi(y)\|_p^p \geq \sum_{n=1}^{k-1} \|\varphi_n(x) - \varphi_n(y)\|_p^p \geq (k-1) \left(\frac{\delta}{2}\right)^p.$$

Thus we can choose $\rho_1(t) = \sum_{n=1}^{\infty} \delta \sqrt[n]{n} \chi_{[S_{n-1}, S_n)}(t)$, $\rho_2(t) = 2t + 1$ and it is clear that Φ is a coarse embedding. \square

G. Yu defined Property A [Yu], which gives a sufficient condition for embeddability of a discrete metric space into a Hilbert space. We recall a characterization of Property A given by J.L. Tu.

Proposition 1 ([Tu]). *A metric space X has property A if and only if for every $R > 0$ and $\varepsilon > 0$ there is a map $\eta : X \rightarrow S(\ell_2(X))$ and $S > 0$ such that*

- (1) $\|\eta(x) - \eta(y)\|_2 \leq \varepsilon$ when $d(x, y) \leq R$;
- (2) $\text{supp } \eta(x) \subset B(x, S)$ for all $x \in X$.

Theorem 2 and the above characterization exhibit the subtle relation between Property A and coarse embeddability.

The following proposition shows that the property of Theorem 3 is not sensitive to changing the index p .

Proposition 2. *Let X have the property described in Theorem 3 with respect to some $1 \leq p < \infty$. Then X has the same property with respect to any $1 \leq q < \infty$*

Proof. For $R > 0$ and $\varepsilon > 0$ given a map $f_p : X \rightarrow S(\ell_p)$ which satisfies conditions (1) and (2) of Theorem 3 define $f_q : X \rightarrow S(\ell_q)$ by the formula

$$f_q(x) = M_{p,q}[f_p(x)],$$

where $M_{p,q} : S(\ell_p) \rightarrow S(\ell_q)$ is the Mazur map.

If $p < q$, by inequalities (1) we have

$$\frac{p}{q} \|f_p(x) - f_p(y)\|_p \leq \|f_q(x) - f_q(y)\|_q \leq C \|f_p(x) - f_p(y)\|_p^{p/q}.$$

Consequently

$$\sup \{ \|f_q(x) - f_q(y)\|_q : x, y \in X, d(x, y) \leq R \} \leq C \varepsilon^{p/q},$$

and

$$\liminf_{S \rightarrow \infty} \{ \|f_q(x) - f_q(y)\|_q : x, y \in X, d(x, y) \geq S \} \geq \frac{p}{q} \delta.$$

The case $p > q$ is proved similarly. \square

In the case of Property A a statement similar to Proposition 2 was studied by Dranishnikov under the name of Property A_p in [Dr].

Corollary 4. *If X admits a coarse embedding into ℓ_2 then it admits a coarse embedding into any ℓ_p with $1 \leq p \leq \infty$. In particular, the separable Hilbert space embeds into all ℓ_p .*

Proof. If X admits a coarse embedding into ℓ_2 then, by Theorem 2, X has the property from Theorem 3 for ℓ_2 . By Proposition 2 has this property also for ℓ_p , $1 \leq p < \infty$ and by Theorem 3 admits an embedding into ℓ_p .

The case $p = \infty$ is clear since ℓ_∞ is a universal space for isometric embeddings. \square

It follows from the above proof that Theorem 3 cannot be extended to a characterization of coarse embeddability into ℓ_p if $p > 2$. Indeed, in that case the procedure described in the above proof would imply that ℓ_p for $p > 2$ embeds coarsely into the Hilbert space, which is not the case by a result of Johnson and Randrianarivony [JR].

In [No] it was shown that $L_p(\mu)$ for $1 \leq p \leq 2$ admit a coarse embedding into the Hilbert space and that coarse embeddability into ℓ_2 is equivalent to coarse embeddability into $L_p[0, 1]$ again for $1 \leq p \leq 2$. This allows us to state

Theorem 5. *Let X be a separable metric space. Then the following conditions are equivalent:*

- (1) X admits a coarse embedding into the Hilbert space;
- (2) X admits a coarse embedding into ℓ_p for some (equivalently all) $1 \leq p < 2$;
- (3) X admits a coarse embedding into $L_p[0, 1]$ for some (equivalently all) $1 \leq p < 2$.

Note that this covers all separable $L_p(\mu)$ -spaces with $1 \leq p \leq 2$. This is particularly interesting since the spaces L_p for different p 's are not coarsely equivalent. To see this assume they are and take $f : L_p(\mu) \rightarrow L_q(\mu)$ to be

the coarse equivalence. Since L_p -spaces are geodesic, f is in fact a quasi-isometry and it induces a Lipschitz equivalence on their ultrapowers. By a theorem of Heinrich [He] ultrapowers of L_p spaces are again L_p spaces (possibly on a different measure), and the assertion follows from a classical fact that Lipschitz equivalence on L_p -spaces induces a linear isomorphism.

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