ON THE ℓ_p-COHOMOLOGY OF GROUPS WITH INFINITE CENTER

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ABSTRACT. We consider the representation of a group on its ℓ_p-cohomology and use it to give a new proof of a theorem of Martin and Valette on the vanishing of the reduced ℓ_p-cohomology for groups with infinite center. We then extend this vanishing theorem to ℓ_p-cohomology with coefficients in reflexive bounded Banach modules. We show that this implies the vanishing of the reduced cohomology of the group, with coefficients in a large class of bounded reflexive Banach modules.

ℓ_p-cohomology is a cohomology theory for non-compact spaces and discrete groups, where certain ℓ_p-summability conditions are imposed on cocycles. It is a quasi-isometry invariant for a large class of well-behaved spaces. The case $p = 2$ is well-studied, however for $p \neq 2$ the absence of tools native to the case of ℓ_2, such as the von Neumann dimension, presents significant difficulties. We refer to [2] for an overview of this topic.

Here we consider a phenomenon considered earlier by several authors, and proved by Martin and Valette [3], that the reduced ℓ_p-cohomology of a group with infinite center vanishes in degree 1. The first instance of this statement was proved by Gromov in [2, Corollary on page 221], who showed that a uniformly contractible smooth manifold, admitting an isometry which is also a translation and has unbounded orbits, has vanishing reduced ℓ_p-cohomology.

In [3] the authors used the correspondence between the reduced ℓ_p-cohomology of Γ and the reduced 1-cohomology with coefficients in the regular representation on $\ell_p(\Gamma)$, as well as p-harmonic functions. Below we provide a more direct argument, which allows to treat a more general case at once, of the reduced ℓ_p-cohomology with coefficients in a bounded, reflexive Banach module, see Definition 1.1.

Theorem 0.1. *Let Γ be a finitely generated group, E a bounded Banach Γ-module and $1 < p < \infty$. If the center $\mathscr{C}(\Gamma)$ of Γ is infinite then $\overline{H}^1_{(p)}(\Gamma, E) = 0$.*

Our method of proof is to use the representation of the group on its cohomology and to show that invariant cohomology classes are represented by invariant cocycles. However, for cocycles, being invariant with respect to an infinite group is not compatible with ℓ_p-summability. This approach is motivated by certain properties of the transfer map in group cohomology. Recall

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that given a group G and its finite index normal subgroup $\Gamma \subseteq G$, the G/Γ-invariant classes in $H^n(\Gamma,\mathbb{F})$ with coefficients in a field $\mathbb{F}$ are precisely those classes, which are in the the image of the transfer map from $H^n(G,\mathbb{F})$.

We also show that the above theorem implies the vanishing of the reduced cohomology groups of Γ with coefficients in a large class of uniformly bounded representation, namely representations on $\ell_p(\Gamma,E)$ for bounded Banach modules E.

After this work was completed a more general version of Theorem 0.1 appeared independently in [1]. Both approaches make use of the Ryll-Nardzewski fixed point theorem.

1. ℓ_p-cohomology with coefficients in Banach modules

Let Γ be an infinite group generated by a finite set $S = S^{-1}$. Let E be a reflexive bounded Banach module; that is, a reflexive Banach space equipped with a uniformly bounded representation π of Γ. Denote by $\mathscr{F}(\Gamma,E)$ the space of functions on Γ taking values in E. The representation ρ of Γ on $\mathscr{F}(\Gamma,E)$ is defined as follows:

$$(\rho_g f)(x) = \pi_g f(xg).$$

Let

$$\ell_p(\Gamma,E) = \left\{ f \in \mathscr{F}(\Gamma,E) : \sum_{\gamma} \|f(\gamma)\|_E^p < \infty \right\},$$

equipped with the norm

$$\|f\|_p = \left(\sum_{\gamma} \|f(\gamma)\|_E^p \right)^{1/p}.$$

The space of p-Dirichlet, E-valued functions on Γ is defined as

$$D_p(\Gamma,E) = \left\{ f \in \mathscr{F}(\Gamma,E) : f - \rho_s f \in \ell_p(\Gamma,E) \text{ for every } s \in S \right\}$$

for every generator $s \in \Gamma$. The space $\mathbb{D}_p(\Gamma,E) = D_p(\Gamma,E)/\mathbb{E}$, where $\mathbb{E}$ denotes those functions in $\mathscr{F}(\Gamma,E)$, that satisfy

$$f = \rho_\gamma f,$$

for every $\gamma \in \Gamma$, can be equipped with the norm

$$\|f\|_{\mathbb{D}} = \left(\sum_{s \in S} \|f - \rho_s f\|_p^p \right)^{1/p}.$$

(The left hand side can be viewed as the discrete gradient of f on Γ, equipped with the left invariant metric.) The Banach space $\ell_p(\Gamma,E)$ is naturally included in $\mathbb{D}_p(\Gamma,E)$. Note that in the case $E = \mathbb{R}$, the image of $\ell_p(\Gamma)$ is closed in $\mathbb{D}_p(\Gamma)$ if and only if Γ is non-amenable.

Definition 1.1. *The reduced ℓ_p-cohomology of degree 1 of the group Γ with coefficients in E is the Banach space $\overline{H}^1_{(p)}(\Gamma,E) = D_p(\Gamma,E) \Big/ \overline{\ell_p(\Gamma,E)} + \mathbb{E}$.*

The representation of Γ on its cohomology. For a group Γ, the center will be denoted $\mathscr{C}(\Gamma)$. The group Γ acts on $\mathbb{D}_p(\Gamma,E)$ via the representation ρ. Observe that

$$\begin{aligned}\|\rho_\gamma f\|_{\mathbb{D}}^p &= \sum_{s\in S}\|\rho_\gamma f-\rho_s\rho_\gamma f\|_p^p\\ &\le C\sum_{s\in S}\sum_{x\in\Gamma}\|f(x)-\pi_{\gamma^{-1}s\gamma}f(x\gamma^{-1}s\gamma)\|_E^p,\end{aligned}$$

for every $\gamma\in\Gamma$, where $C=\sup_{\gamma\in\Gamma}\|\pi_\gamma\|$. This yields

Lemma 1.2. *There exists $C>0$ such that $\|\rho_\gamma\|<C$, as an operator on $\mathbb{D}_p(\Gamma,E)$, for every $\gamma\in\mathscr{C}(\Gamma)$.*

Since ρ preserves $\ell_p(\Gamma,E)$, it induces a representation ρ^* of the group Γ on $\overline{H}^1_{(p)}(\Gamma,E)$. By the definition of $\overline{H}^1_{(p)}(\Gamma,E)$, we have the following

Corollary 1.3. *The cohomology representation ρ^* of Γ on $\overline{H}^1_{(p)}(\Gamma,E)$ is trivial.*

2. Proof of Theorem 0.1

The thrust of the proof of the Theorem is the fact that ρ^*-invariant classes have to be represented by $\mathscr{C}(\Gamma)$-invariant cocycles, but that last condition is incompatible with ℓ_p-type conditions.

Proof of Theorem 0.1. If $\gamma\in\mathscr{C}(\Gamma)$, the center of Γ, then we have that the norm of ρ_γ is uniformly bounded in γ on $\mathbb{D}_p(\Gamma,E)$. Consequently,

$$b_\gamma=\rho_\gamma f-f$$

is a bounded 1-cocycle $\mathscr{C}(\Gamma)\to\ell_p(\Gamma,E)$ for every $f\in\mathbb{D}_p(\Gamma,E)$. For each such f there exists an element $m_f\in\overline{\ell_p(\Gamma,E)}^{\mathbb{D}}$ satisfying

$$\rho_\gamma m_f-m_f=\rho_\gamma f-f$$

for every $\gamma\in\mathscr{C}(\Gamma)$. Indeed, the space $\mathbb{D}_p(\Gamma,E)$ can be identified with a closed subspace of the space $\ell_p(\mathscr{E},E)$, where $\mathscr{E}$ is the set of edges of the Cayley graph associated to the given generating set. Since $1<p<\infty$, as a closed subspace of a reflexive space, $\overline{\ell_p(\Gamma,E)}^{\mathbb{D}}$ is reflexive. It is a standard consequence of the Ryll-Nardzewski fixed point theorem [4], together with passing to the equivalent norm

$$\|v\|'=\sup_{\gamma\in\Gamma}\|\pi_\gamma v\|_E,$$

on E that any bounded cocycle from a discrete group into a reflexive Banach space is a coboundary.

Consequently, the element $f-m_f\in\mathbb{D}_p(\Gamma,E)$ is $\mathscr{C}(\Gamma)$-invariant:

$$(f-m_f)=\rho_\gamma(f-m_f)$$

$\gamma\in\mathscr{C}(\Gamma)$ and

$$[f-m_f]=[f],$$

as classes in $\overline{H}^1_{(p)}(\Gamma,E)$.

However, if an element $\alpha \in \mathbb{D}_p(\Gamma,E)$ is invariant under the action of $\mathscr{C}(\Gamma)$ then it represents the zero class in $\overline{H}^1_{(p)}(\Gamma,E)$. Indeed, let $h \in \Gamma$. Then

$$\begin{aligned} \|\alpha\|^p_{\mathbb{D}} &\geq \|\alpha - \rho_s\alpha\|^p_p \\ &\geq \sum_{\gamma\in\mathscr{C}(\Gamma)} \|\alpha(h\gamma) - \pi_s\alpha(h\gamma s)\|^p_E \\ &= \sum_{\gamma\in\mathscr{C}(\Gamma)} \left\|\pi_{\gamma^{-1}}(\alpha(h) - \pi_s\alpha(hs))\right\|^p_E \\ &\geq \frac{1}{C}\sum_{\gamma\in\mathscr{C}(\Gamma)} \|\alpha(h) - \pi_s\alpha(hs)\|^p_E, \end{aligned}$$

where $C = \sup_\gamma \|\pi_\gamma\|$. The infinite sum on the right hand side is thus finite if and only if $\alpha(h) = \pi_s\alpha(hs) = \rho_s\alpha(h)$ for every $s \in S$. It follows that $\alpha = \rho_s\alpha$ for every $s \in S$. □

Vanishing of the reduced 1-cohomology of Γ. Let E be a Banach Γ-module with respect to a representation π. A cocycle for π is a function $b : \Gamma \to E$ satisfying $b(gh) = \pi_g b(h) + b(g)$. The space $Z^1(\Gamma,\pi)$ of π-cocycles is equipped with the topology of uniform convergence on compact subsets of Γ. A cocycle b is a coboundary if there exists $v \in E$ such that $b(g) = \pi_g v - v$ for all $g \in G$. The coboundaries form a subspace of $Z^1(G,\pi)$, denoted $B^1(G,\pi)$. The reduced 1-cohomology of Γ with coefficients in a G-module E is the vector space $\overline{H}^1(\Gamma,\pi) = Z^1(\Gamma,\pi)/\overline{B^1(\Gamma,\pi)}$. We will now show that the vanishing of $\overline{H}^1_p(\Gamma,E)$ implies the vanishing of certain reduced 1-cohomology groups.

Lemma 2.1. *Let $b : \Gamma \to \mathscr{F}(\Gamma,E)$ be a cocycle associated to the representation ρ, introduced earlier. Then there exists $f \in \mathscr{F}(\Gamma,E)$ such that $b_\gamma = \rho_\gamma f - f$.*

Proof. Given b, define $f(\gamma) = \pi_\gamma^{-1} b_\gamma(e)$. Then it is easy to verify that f satisfies the required condition. □

Corollary 2.2. *Let Γ be a finitely generated group with infinite center, E be a bounded reflexive Banach Γ-module, ρ be the uniformly bounded representation on $\ell_p(\Gamma,E)$ introduced earlier and $1 < p < \infty$. Then $\overline{H}^1(\Gamma,\ell_p(\Gamma,E)) = 0$.*

Proof. Let $b : \Gamma \to \ell_p(\Gamma,E)$ be a cocycle for ρ. By the previous lemma, $b_\gamma = \rho_\gamma f - f$ for some f, which by definition is an element of $D_p(\Gamma,E)$. Thus f is a limit of elements of $f_i \in \ell_p(\Gamma,E)$ in $D_p(\Gamma,E)$, by Theorem 0.1. This convergence, however, is the same as the convergence of $\rho_\gamma f_i - f_i$ to b, uniformly on compact subsets of Γ. □

References

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