$\mathrm{Aut}(\mathbb{F}_5)$ has property (T)

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(joint work with M. Kaluba and N. Ozawa)

Property (T) is a fundamental rigidity property for groups. It was introduced by Kazhdan in 1966. The simplest way to define it is through one of its characterizations: a finitely generated group G has property (T) if every action of G on a Hilbert space by affine isometries has a fixed point.

Property (T) is now known to hold for a few classes of groups, nevertheless it should certainly be considered a rare property. There are two main reasons behind property (T). The first is an algebraic structure of the group rigid enough to force this kind of behavior. This is the case for $SL_n(\mathbb{Z})$, $n \geq 3$, and more generally for higher rank Lie groups and their lattices. The second is a spectral property, that can be described as possesing positive curvature with respect to Euclidean geometry. More precisely, the condition is that the link of a generating set, or of a contractible simplicial complex on which G acts cocompactly, has the first positive eigenvalue greater than 1/2. This method yields property (T) for automorphism groups of thick buildings, such as $\widetilde{A}_2$ buildings, and certain random hyperbolic groups in the Gromov density model. We refer to [1] for an excellent overview of the topic.

The aim of our work is to prove property (T) for a new group. Denote by $\mathrm{Aut}(\mathbb{F}_\mathrm{n})$ the group of automorphisms of the free group $\mathbb{F}_n$ on n generators, and by $\mathrm{Out}(\mathbb{F}_n)$ its quotient by the subgroup of inner automorphisms. The abelianization $\alpha : \mathbb{F}_n \to \mathbb{Z}^n$ induces a surjection

$$\alpha_* : \mathrm{Aut}(\mathbb{F}_n) \to \mathrm{GL}_n(\mathbb{Z}),$$

which factors through $\mathrm{Out}(\mathbb{F}_n)$. For $n = 2$, the map $\mathrm{Out}(\mathbb{F}_2) \to \mathrm{GL}_2(\mathbb{Z})$ is in fact an isomorphism. The special automorphism group $\mathrm{SAut}(\mathbb{F}_n) \subseteq \mathrm{Aut}(\mathbb{F}_n)$ is the preimage $\alpha_*^{-1}(\mathrm{SL}_n(\mathbb{Z}))$. This subgroup has index 2 in $\mathrm{Aut}(\mathbb{F}_n)$ and has a particularly convenient set of generators within $\mathrm{Aut}(\mathbb{F}_n)$: it is generated by the Nielsen transformations. We fix this generating set for the group $\mathrm{SAut}(\mathbb{F}_n)$. Our main result is the following

Theorem 1 ([4])**.** *The group* $\mathrm{SAut}(\mathbb{F}_5)$ *has property* (T) *with Kazhdan constant* $\kappa \geq 0.176$.

The proof relies on the following characterization of property (T) due to Ozawa.

Theorem 2 ([6])**.** *A finitely generated group has property* (T) *if and only if there exists a finite collection* $\xi_i \in \mathbb{R}G$ *and* $\kappa > 0$ *such that*

$$\Delta^2 - \kappa\Delta = \sum_{i=1}^{n} \xi_i^* \xi_i. \tag{1}$$

The condition (1) can easily be translated into the existence of a positive definite matrix P, indexed by some finite subset $E \subseteq G$, such that

$$\Delta^2 - \kappa\Delta = \mathbf{x}P\mathbf{x}^T, \tag{2}$$

where $\mathbf{x} = [\delta_{g_1}, \dots, \delta_{g_n}]$ and the g_i run through the elements of E.

From now on we fix E to be the ball of radius 2 in the word length metric on G. The strategy to prove Theorem 1 is to find a solution the equation (2) with the assistance of a computer. There are solvers appropriate for semidefinite programming, i.e. software designed to solve systems of linear equations such as (2), with the restriction that P is positive semidefinite. This approach was used successfully to reprove property (T) for $SL_n(\mathbb{Z})$ by Netzer and Thom for $n = 3$ [5], Fujiwara and Kabaya for $n = 3, 4$ [2] and Kaluba and Nowak for $n = 3, 4, 5$ [3]. In these cases the new proof included new, drastically improved, estimates of Kazhdan constants for $\mathrm{SL}_n(\mathbb{Z})$, $n = 3, 4, 5$.

In the case of $\mathrm{Aut}(\mathbb{F}_5)$ the ball $B(e, 2)$ has 4 641 elements, and consequently the matrix P depends on 10 771 761 variables, too many for a solver to handle. In order to simplify the problem and reduce the number of variables we symmetrize the problem. Consider an action of a finite group Σ of automorphisms of G which preserves the set E and the Laplacian Δ. We show that if a solution P of (2) exists then there exists a solution P invariant under the action of Σ, and thus it suffices to find such a Σ-invariant P. This significantly reduces the number of variables in the matrix, however poses a new problem: solvers are not able to handle group-invariant matrices.

A classical theorem of Wedderburn implies that the algebra of Σ-invariant matrices is isomorphic to a direct sum of matrix algebras. The next step is thus a construction of an explicit isomorphism

$$\mathbb{M}_E^{\Sigma} \simeq \oplus_{\pi \in \hat{\Sigma}} 1_{\dim \pi} \otimes \mathbb{M}_{m_\pi},$$

where $\hat{\Sigma}$ is the unitary dual of Σ and m_π denotes the multiplicity of $\pi \in \hat{\Sigma}$. For $G = \mathrm{Aut}(\mathbb{F}_5)$ we choose $\Sigma = \mathbb{Z}_2 \wr S_5$ and the above isomorphism is constructed using a system of minimal projections. The resulting reduction in complexity is indeed significant: from the previously mentioned 10 771 761 variables to 13 232 variables in 36 blocks. Then using a solver we obtain a solution P with accuracy of the order 10^{-9} and $\kappa = 1.2$.

The last and crucial step is the certification of the solution. The solution provided by the solver is by definition approximate. However, since the solution is obtained with very good accuracy an additional argument, based on the fact that the Laplacian Δ is an order unit for self-adjoint elements of the augmentation ideal in $\mathbb{R}G$, allows to deduce the existence of a mathematically precise solution. This argument turns the above reasoning into a rigorous proof of property (T) for $\mathrm{Aut}(\mathbb{F}_5)$ and in the process gives an explicit estimate on the Kazhdan constant of G.

References

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