Kazhdan projections in Banach spaces

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A Kazhdan projection is a central idempotent p in the maximal group C^*-algebra $C^*_{\max}(G)$ of G such that for every unitary representation π of G on a Hilbert space H the image $\pi(p) \in B(H)$ is the orthogonal projection for H onto the subspace of invariant vectors $H^\pi \subseteq H$. It was proved by Akemann and Walters [1] that the existence of a Kazhdan projection characterizes Kazhdan's property (T). Another proof of this fact was given by Valette [7]. Kazhdan projections have many applications, in particular they are the source of the few known counterexamples to certain versions of the Baum-Connes conjecture. The reason is that K-theory classes represented by projections of Kazhdan-type usually do not live in the image of the Baum-Connes assembly map. Nevertheless, Kazhdan projections have been considered mysterious objects and no explicit constructions of such projections were known.

Our main result is a new method to construct Kazhdan projections in the general setting of uniformly convex Banach spaces. The main tools are random walks and the associated Markov operators, for which we provide new estimates and convergence results. For a locally compact group G consider an (linear) isometric representation π on a uniformly convex Banach space E. The subspace of invariant vectors E^π has a natural π-invariant complement E_π, so that $E = E_\pi \oplus E^\pi$ [2, 3]. The Kazhdan constant of π relative to a compact generating set S is the number $\inf_{v\in E_\pi} \sup_{s\in S} \|\pi_s v - v\|$ and we say that π has a spectral gap if this constant is positive. For a certain large class of admissible probability measures μ on G we consider the Markov operator $A^\mu_\pi = \int_G \pi_g v d\mu$.

We first explain the quantitative relation between projections onto invariant vectors and spectral gaps in the setting of uniformly convex spaces and in particular prove

Theorem 1. *Let μ be an admissible measure. If a representation π as above has a positive Kazhdan constant then the restriction of the Markov operator to E_π satisfies $\|A^\mu_\pi|_{E_\pi}\| \leq \lambda < 1$, where λ depends only on the Kazhdan constant of π, the modulus of uniform convexity of E and the measure μ.*

Moreover, the projection $\mathcal{P}_\pi : E \to E^\pi$ along E_π is given by the formula

$$\mathcal{P}_\pi = I - \left(\sum_{n=0}^{\infty} (A^\mu_\pi)^n\right)(I - A^\mu_\pi).$$

and $\mathcal{P}_\pi = \lim_{n\to\infty} (A^\mu_\pi)^n$, where the convergence is uniform for a family $\mathcal{F}$ of isometric representations as long as λ above is uniform for all representations in $\mathcal{F}$.

A quantitative converse to the above holds as well. From the above theorem we derive an explicit construction of Kazhdan projections in various group Banach algebras. Let $\mathcal{F}$ be a family of isometric representations of G on a uniformly convex family of Banach spaces. Let $C_c(G)$ denote the convolution algebra of compactly supported continuous functions on G. For $f \in C_c(G)$ define

$$\|f\|_{\mathcal{F}} = \sup_{\pi \in \mathcal{F}} \|\pi(f)\|$$

and let $C_{\mathcal{F}}(G)$ be the Banach algebra obtained as a completion of $C_c(G)$ in the above norm. A Kazhdan projection in $C_{\mathcal{F}}(G)$ is then a central idempotent $p \in C_{\mathcal{F}}(G)$ such that $\pi(p) = \mathcal{P}_\pi$ for every $\pi \in \mathcal{F}$.

Theorem 2. *There exists a Kazhdan projection in $C_{\mathcal{F}}(G)$ if and only if there is a uniform positive lower bound on the Kazhdan constants for all $\pi \in \mathcal{F}$.*

This construction of Kazhdan projections is new in particular in the setting of Hilbert space and property (T), where $\mathcal{F}$ is taken to be the collection of all unitary representations of G. It also allows to give a direct comparison of various versions of property (T) in the context of Banach spaces: properties (TE), FE studied in [2] and Lafforgue's reinforced Banach property (T), introduced in [5].

Kazhdan projections can be viewed as invariant means in the setting of property (T). We apply them to show a natural generalization of property (τ) to the context of a uniformly convex Banach space E and show that it is equivalent to the fact that the related family of Cayley graphs of finite quotients forms a family of E-expanders. We also obtain results in ergodic theory, where we apply Kazhdan projections to a question posed by Kleinbock and Margulis on shrinking target problems.

Finally, we show a new construction of non-compact ghost projections for warped cones, a class of metric spaces constructed by Roe using an action of group on a compact space [6]. Ghosts are certain operators on Hilbert modules, that are locally invisible at infinity, yet are not compact. Such ghost projections are known to give rise to K-theory classes that are obstructions to the coarse Baum-Connes conjecture. Their existence was previously established only for expanders, Willett and Yu asked for new examples.

Theorem 3. *Let G be a finitely generated group acting ergodically on a compact metric probability space M by measure preserving Lipschitz homeomorphisms. If the corresponding unitary representation of G on $L_2(M)$ has a spectral gap then the warped cone $\mathcal{O}_G(M)$ has a non-compact ghost projection, which is a limit of finite propagation operators.*

We conjecture that the coarse Baum-Connes conjecture fails for warped cones provided by the above theorem. As a particular example consider the action of certain free subgroups $G = \mathbb{F}_2$ on $M = SU(2)$. The spectral gap property for many such subgroups was established by Bourgain and Gamburd [4].

In relation to the above theorem Guoliang Yu posed the following question: *is there an action with a spectral gap such that the corresponding warped cone does not contain a coarsely embedded sequence of expanders?*

References

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