

Diameters, distortion and eigenvalues

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Abstract

We study the relation between the diameter, the first positive eigenvalue of the discrete p -Laplacian and the ℓ_p -distortion of a finite graph. We prove an inequality relating these three quantities and apply it to families of Cayley and Schreier graphs. We also show that the ℓ_p -distortion of Pascal graphs, approximating the Sierpinski gasket, is bounded, which allows to obtain estimates for the convergence to zero of the spectral gap as an application of the main result.

Keywords: ℓ_p -distortion; p -Laplacian; spectral gap; algebraic connectivity

1. Introduction

Distortion of a finite metric space X is, roughly speaking, a measure of how well can X be embedded into the Hilbert space or, more generally, a Banach space. The study of distortion of finite metric spaces has a long history, and in recent years the subject received some attention, in part due to applications in theoretical computer science, where low distortion embeddings imply good computational properties [3, 31, 32].

Another area of research which is on the border of discrete mathematics, probability, algebra and computer science is the study of spectral, asymptotic and combinatorial properties of infinite families of graphs of constant, or uniformly bounded, degree. A characteristic which is often at the heart of such studies is the first non-zero eigenvalue of the discrete Laplacian, known also as spectral gap.

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Various relations between the spectral gap and other natural characteristics of a finite graph are known, we refer to [15] for a list of such relations. Two notable inequalities of this type are due to Alon and Milman [1] and Chung [11]. In this paper we study a similar relation, additionally involving the ℓ_p -distortion of a finite graph.

Our first result and the main tool in subsequent considerations is a general inequality relating the diameter, the first non-zero eigenvalue of the discrete p -Laplacian and the ℓ_p -distortion of a family of graphs (Theorem 3). In order to prove it we introduce a certain mild regularity condition, called volume distribution. This condition guarantees that in some sense, sets of large volume cannot be small in diameter. It is of particular importance for infinite families of graphs, where we require uniform behavior of the volume distribution. This condition is however natural and it is satisfied by many families of graphs, examples of which are provided.

We apply the inequality to Cayley graphs of finite groups and families of Schreier graphs with the set of vertices of the form G/H_n , where G is a residually finite group generated by a finite set and $\{H_n\}_{n \in \mathbb{N}}$ is a sequence of finite index subgroups of G with trivial intersection. The case of special interest is when the group G is a finitely generated group of automorphisms of a regular rooted tree and $H_n = \text{st}_G(u_n)$ is a stabilizer of a vertex u_n of the n -th level, belonging to an infinite geodesic ray joining the root with infinity. The study of spectral properties of such graphs was initiated in [4, 23, 24, 27] and remains an active field of research.

One such family is obtained from the Hanoi Towers groups on 3 pegs. These graphs, called Pascal graphs, were studied in [25, 35] and geometrically approximate the Sierpinski gasket. Our second result is that the family of Pascal graphs has uniformly distributed volume (Theorem 10) and bounded ℓ_p -distortion (Theorem 11). As a consequence of this fact and our inequality we obtain exponential convergence to zero for the spectral gap of the discrete p -Laplacian. In the special case $p = 2$ such convergence follows from [25]. In the process we also show that information on distortion can give better estimates for convergence of the spectral gap than some previously known inequalities.

It is worth mentioning that not all graphs having a fractal structure in the limit have bounded distortion. Examples are furnished by the diamond graphs and Laakso graphs, which are constructed using recursive substitution and whose ℓ_p -distortion is unbounded for any $p > 1$ [29].

Finally, in the last section, we discuss the remaining problems.

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2. Definitions

2.1. Distortion

Let X be a finite metric space, \mathcal{E} be a Banach space and let $f : X \rightarrow \mathcal{E}$ be a one-to-one map which is 1-Lipschitz:

$$\|f(x) - f(y)\|_{\mathcal{E}} \leq d(x, y),$$

for all $x, y \in X$. Let

$$L_f = \max_{x, y \in X} \frac{d(x, y)}{\|f(x) - f(y)\|_{\mathcal{E}}}. \quad (1)$$

Definition 1. *The \mathcal{E} -distortion of X is the number*

$$c_{\mathcal{E}}(X) = \inf L_f, \quad (2)$$

where the infimum is taken over all $f : X \rightarrow \mathcal{E}$ which are one-to-one and 1-Lipschitz.

Intuitively, distortion measures the most efficient way to embed X into \mathcal{E} . By $c_{(p)}(X)$ we will denote the distortion of X for $\mathcal{E} = \ell_p$. In the case $p = 2$ the number $c_{(2)}(X)$ is called the *Euclidean distortion* of X . Euclidean distortion is particularly well-studied, see for instance [31, 32, 33] and the references therein.

Bourgain [9] showed that Euclidean distortion of any finite metric space X satisfies $c_{(2)}(X) \leq C_B \log |X|$, where $|X|$ denotes the cardinality of X and C_B is a universal constant. This bound is sharp and is realized by expander graphs (that is, infinite family of finite graphs of uniformly bounded degree with spectral gap uniformly bounded away from zero) [31].

2.2. The discrete p -Laplacian and its eigenvalues

Let $\Gamma = (V, E)$ be a finite, undirected, connected graph with vertex set V and edge set E . We write $x \sim y$ to denote the fact that vertices x and y are joined by an edge. We allow loops and multiple edges and by $\omega(x, y)$ we denote the number of edges linking vertices x and y . The set of vertices of Γ can be viewed as a metric space when equipped with the combinatorial metric.

As usual, given a set S , by $\ell_p(S)$ we denote the space of function $f : S \rightarrow \mathbb{R}$ for which the p -norm $\|f\|_p = (\sum_{x \in S} |f(x)|^p)^{1/p}$ is finite. The symbol 1_A will denote the characteristic function of a set A .

Let $1 < p < \infty$. The p -Laplacian Δ_p is an operator $\Delta_p : \ell_p(V) \rightarrow \ell_p(V)$, defined by the formula

$$\Delta_p f(x) = \sum_{x \sim y} (f(x) - f(y))^{[p]} \omega(x, y),$$

for $f : V \rightarrow \mathbb{R}$, where $a^{[p]} = |a|^{p-1} \text{sign}(a)$. For $p \neq 2$ the p -Laplacian is a non-linear operator, while for $p = 2$ it is the standard linear discrete Laplacian. The p -Laplacian is a well-known operator in the study of partial differential equations, for graphs it was considered in e.g. [2, 7, 10, 37].

A real number λ is an eigenvalue of the p -Laplacian Δ_p if there exists a function $f : V \rightarrow \mathbb{R}$ such that

$$\Delta_p f = \lambda |f|^{p-1} f.$$

The eigenvalues of the p -Laplacian are quite difficult to compute in the case $p \neq 2$, due to non-linearity of Δ_p . Define

$$\lambda_1^{(p)}(\Gamma) = \inf \left\{ \frac{\sum_{x \in V} \sum_{y \sim x} |f(x) - f(y)|^p \omega(x, y)}{\inf_{\alpha \in \mathbb{R}} \sum_{x \in V} |f(x) - \alpha|^p} \right\} \quad (3)$$

with the infimum taken over all $f : V \rightarrow \mathbb{R}$ such that f is not constant. It was proved in [10], by means of the variational principle, that $\lambda_1^{(p)}$ is the smallest positive eigenvalue of the discrete p -Laplacian Δ_p or the p -spectral gap. For $p = 2$ this agrees with the definition of the first positive eigenvalue as the minimizer of the Rayleigh quotient $\langle \Delta_2 f, f \rangle / \langle f, f \rangle$ over all f which are orthogonal to the constant functions on V , since for $f : V \rightarrow \mathbb{R}$ satisfying $\langle f, 1_V \rangle = 0$ we have

$$\|f\|_2^2 = \inf_{\alpha \in \mathbb{R}} \|f - \alpha 1_V\|_2^2.$$

In graph theory $\lambda_1^{(2)}$ is often referred to as algebraic connectivity, see for instance [15].

For $p = 1$ the definition of $\lambda_1^{(1)}$ alone still makes sense and we will use state our results for $p \geq 1$, even though the interpretation of $\lambda_1^{(p)}$ as an eigenvalue of Δ_p is valid only for $p > 1$.

3. Distortion and the spectral gap

Given a finite metric space X we denote by $\delta(X)$ its diameter and by $|X|$ its cardinality. We start by defining a constant ρ_ϵ which describes certain geometric features of X .

Definition 2. Let X be a finite metric space. Given $0 < \epsilon < 1$ define the constant $\rho_\epsilon(X) \in [0, 1]$, called the volume distribution, by the relation

$$\rho_\epsilon(X) = \min \left\{ \frac{\delta(A)}{\delta(X)} : A \subseteq X \text{ such that } |A| \geq \epsilon |X| \right\}.$$

When $X = V$ is the vertex set of a finite graph Γ , we use the notation $\rho_\epsilon(\Gamma) = \rho_\epsilon(V)$.

The following theorem gives the inequality between the p -spectral gap, diameter and the ℓ_p -distortion.

Theorem 3. Let Γ be a finite, connected graph of degree bounded by k and let $1 \leq p < \infty$. Then for every $0 < \epsilon < 1$,

$$\lambda_1^{(p)}(\Gamma) \leq C \left(\frac{c_{(p)}(\Gamma)}{\delta(\Gamma)} \right)^p,$$

where $C = C(k, \epsilon, \rho_\epsilon, p) = \frac{k}{1 - \epsilon} \left(\frac{2}{\rho_\epsilon(\Gamma)} \right)^p$.

Proof of Theorem 3. Let $\{e_n\}_{n \in \mathbb{N}}$ be the standard basis vectors in $\ell_p(\mathbb{N})$ and let $F : \Gamma \rightarrow \ell_p(\mathbb{N})$, where $F(x) = \sum_{n \in \mathbb{N}} e_n F_n(x)$ and $F_n : V \rightarrow \mathbb{R}$ are coordinate functions, be a one-to-one, 1-Lipschitz map. Since

$$\inf_{\alpha \in \mathbb{R}} \left| \|F_n\|_p - \|\alpha 1_V\|_p \right| \leq \inf_{\alpha \in \mathbb{R}} \|F_n - \alpha 1_V\|_p \leq \|F_n\|_p,$$

we conclude that for every $n \in \mathbb{N}$ there exists $\alpha_n \in \mathbb{R}$ such that

$$\inf_{\alpha \in \mathbb{R}} \|F_n - \alpha 1_V\|_p^p = \|F_n - \alpha_n 1_V\|_p^p,$$

and moreover,

$$|\alpha_n| \leq \frac{2\|F_n\|_p}{\|1_V\|_p} = \frac{2\|F_n\|_p}{|V|^{1/p}}. \quad (4)$$

We have $\sum_{n \in \mathbb{N}} |\alpha_n|^p < \infty$. Indeed, inequality (4) yields

$$\sum_{n \in \mathbb{N}} |\alpha_n|^p \leq \frac{2^p}{|V|} \sum_{n \in \mathbb{N}} \sum_{x \in V} |F_n(x)|^p = \frac{2^p}{|V|} \sum_{x \in V} \|F(x)\|_p^p < \infty.$$

By virtue of the above estimate, the vector $w = \sum_{n \in \mathbb{N}} e_n \alpha_n$ is an element of ℓ_p and we can define a new embedding $f : V \rightarrow \ell_p$ by shifting by w : $f(x) = \sum_{n \in \mathbb{N}} e_n f_n(x)$, where

$$f_n(x) = F_n(x) - \alpha_n.$$

Then for every $n \in \mathbb{N}$ the inequality

$$\lambda_1^{(p)} \sum_{x \in V} |f_n(x)|^p \leq \sum_{x \in V} \sum_{y \sim x} |f_n(x) - f_n(y)|^p \omega(x, y), \quad (5)$$

holds, by the definition of $\lambda_1^{(p)} = \lambda_1^{(p)}$ and the choice of α_n . Then, applying (5) coordinate-wise, we have

$$\begin{aligned} \sum_{x \in V} \|f(x)\|_p^p &= \sum_{n \in \mathbb{N}} \sum_{x \in V} |f_n(x)|^p \\ &\leq \frac{1}{\lambda_1^{(p)}} \sum_{n \in \mathbb{N}} \sum_{x \in V} \sum_{y \sim x} |f_n(x) - f_n(y)|^p \omega(x, y) \\ &\leq \frac{1}{\lambda_1^{(p)}} \sum_{x \in V} \sum_{y \sim x} \|f(x) - f(y)\|_p^p \omega(x, y) \\ &\leq \frac{1}{\lambda_1^{(p)}} \sum_{x \in V} \sum_{y \sim x} \omega(x, y) \\ &\leq \frac{k|V|}{\lambda_1^{(p)}}, \end{aligned}$$

since F is 1-Lipschitz and, therefore, $\|f(x) - f(y)\|_p = \|F(x) - F(y)\|_p \leq 1$ whenever $x \sim y$. Thus at least $\epsilon|V|$ of $x \in V$ satisfy

$$\|f(x)\|_p \leq \left(\frac{k}{(1 - \epsilon)\lambda_1^{(p)}} \right)^{1/p}.$$

Consequently, we can find two points $x_0, y_0 \in V$ such that

$$d(x_0, y_0) \geq \rho_\epsilon(\Gamma)\delta(\Gamma)$$

and at the same time

$$\|f(x_0) - f(y_0)\|_p \leq 2 \left(\frac{k}{(1-\epsilon)\lambda_1^{(p)}} \right)^{1/p}.$$

For the constant L_F we obtain

$$\begin{aligned} L_F &= \max_{x, y \in V} \frac{d(x, y)}{\|F(x) - F(y)\|_p} \\ &\geq \frac{d(x_0, y_0)}{\|f(x_0) - f(y_0)\|_p} \\ &\geq \frac{1}{2} \rho_\epsilon(\Gamma)\delta(\Gamma) \cdot \left(\frac{k}{(1-\epsilon)\lambda_1^{(p)}} \right)^{-1/p} \\ &\geq \frac{1}{2} \rho_\epsilon(\Gamma)\delta(\Gamma) \left(\lambda_1^{(p)} \right)^{1/p} \left(\frac{k}{1-\epsilon} \right)^{-1/p}. \end{aligned}$$

Since the right hand side of the inequality is independent of F we pass to the infimum over all $F : \Gamma \rightarrow \ell_p$ which are one-to-one and 1-Lipschitz and the assertion follows. \square

3.1. An Alon-Milman-type inequality for the p -spectral gap

Alon and Milman proved in [1] that for every graph $\Gamma = (V, E)$ with degree bounded by k the inequality

$$\delta(\Gamma) \leq 2 \sqrt{\frac{2k}{\lambda_1^{(2)}(\Gamma)}} (\log_2 |V|), \quad (6)$$

holds.

On the other hand, Bourgain proved in [9] that the Euclidean distortion of any finite metric space X satisfies $c_{(2)}(X) \leq C_B \log_2 |X|$ for some universal constant $C_B > 0$. In fact his techniques show that

$$c_{(p)}(X) \leq C_B^{(p)} (\log_2 |X|),$$

for a universal constant $C_B^{(p)}$, depending only on p . This estimate together with Theorem 3 yields a general inequality between the diameter of a graph, number of vertices and $\lambda_1^{(p)}$. For every $0 < \epsilon < 1$ we have

$$\delta(\Gamma) \leq \left(C(k, \epsilon, \rho_\epsilon, p)^{1/p} C_B^{(p)} \right) \frac{\log_2 |V|}{\left(\lambda_1^{(p)}(\Gamma) \right)^{1/p}}. \quad (7)$$

For fixed $p \geq 1$, $\epsilon > 0$ and a family of graphs of degree bounded by k , if $\rho_\epsilon(\Gamma)$ is uniformly bounded away from zero, then the constants $C(k, \epsilon, \rho_\epsilon, p)$ are uniformly bounded above and in that case Theorem 3, together with the result of Bourgain, allows to recover and generalize to any p , the inequality (6), up to a multiplicative constant depending on the family.

4. Schreier graphs and uniform volume distribution

In this section we consider infinite families of Cayley graphs and Schreier graphs of groups of automorphisms of trees (see [4, 20, 24, 34] for background on this topic). We will demonstrate on examples that for sequences of graphs Theorem 3 often gives results asymptotically close to optimal and in some cases gives better results than some previously known inequalities. Recall that given a group G with a finite generating set A and a subgroup $P \leq G$, the Schreier graph $\Gamma = \Gamma(G, P, A)$ consists of the set of vertices being in bijection with left cosets gP and the set of edges $E = \{(gP, agP) : a \in A \cup A^{-1}\}$. The Cayley graph of G is a particular case when $P = \{1\}$. Every regular graph of even degree $2m$ can be represented as a Schreier graph of the free group \mathbb{F}_m with respect to a certain subgroup, see [16, IV.A.15] for a discussion.

We are interested in studying sequences of Schreier graphs of the form $\Gamma_n = (G, P_n, A)$, where $\{P_n\}$ is a descending sequence of finite index subgroups with trivial intersection. In other words, we are considering a covering sequence $\{\Gamma_n\}$ of finite Schreier graphs (i.e., where Γ_{n+1} covers Γ_n). Groups with such chains are residually finite and act naturally and level transitively on infinite, spherically homogeneous rooted trees, where P_n serve as stabilizers of vertices of level n of the rooted trees [6, 19, 20, 34]. We will consider here three examples: finite lamplighter groups, the group of intermediate growth constructed in [17, 18] and the Hanoi Towers group $H^{(3)}$ introduced in [23, 24]. Many more examples of such graphs can be found in [4, 22, 30, 39].

We are interested in three characteristics of such sequences: diameters, λ_1 and distortion. It is easy to see that the asymptotic behavior of diameters and distortion does not depend on the generating set A , however it is not clear if the same is true for λ_1 , therefore we will always indicate the generating set we are using.

4.1. Uniform volume distribution

We now want to consider families of graphs for which the constant $C(k, \epsilon, \rho_\epsilon, p)$ is uniformly bounded above. This amounts to controlling ρ_ϵ and motivates the following notion.

Definition 4. *Given a family of finite metric spaces $\mathcal{X} = \{X_i\}$ and $0 < \epsilon < 1$ we say that the family \mathcal{X} has uniform volume distribution if there exist $\epsilon > 0$ and $c > 0$ such that $\rho_\epsilon(X_i) \geq c$ for every $X_i \in \mathcal{X}$.*

Such families will be of particular interest. Of course not all families of bounded degree graphs have uniform volume distribution.

Example 5. Consider a 3-regular tree T and let $B(n)$ denote a ball of radius n around a fixed point x_0 .

Define the graph Γ_n as $B(n)$ with a path P_n of length $|B(n)|$ attached at x_0 . Then $|B(n)| = |\Gamma_n|/2$, however $\delta(B(n)) = 2n$ and $\delta(\Gamma_n) \geq |P_n| \geq |B(n)|$, where the latter grows exponentially, so that the ratio $\delta(B(n))/\delta(\Gamma_n)$ tends to 0.

Fortunately, many families satisfy the conditions of Definition 4 in a natural way. Recall that a graph is vertex transitive if the group of automorphisms acts transitively on its vertices. Cayley graphs of groups are examples of vertex transitive graphs.

Proposition 6. *Let Γ be a vertex transitive connected graph on at least 3 vertices. Then $\rho_{1/2}(\Gamma) \geq \frac{1}{4}$.*

Proof. By $B_x(r) = \{y : d(x, y) \leq r\}$ we denote the ball or radius r centered at x . If $\delta(\Gamma) \leq 2$ then the claim can be verified directly. Assume that $\delta(\Gamma) \geq 3$. First let us prove that $|B_x(\delta(\Gamma)/4)| < |V|/2$ for any vertex $x \in V$. Assume the contrary. Take a pair of points $x, y \in V$ such that

$$d(x, y) = \delta(\Gamma).$$

We have

$$B_x(\delta(\Gamma)/4) \cap B_y(\delta(\Gamma)/4) = \emptyset$$

and there is a point which does not belong to either of the two balls. Indeed, if there is no such point then

$$\delta(\Gamma) = d(x, y) \leq 2(\delta(\Gamma)/4) + 1 = \delta(\Gamma)/2 + 1,$$

which implies $\delta(\Gamma) \leq 2$. By the triangle inequality and homogeneity we obtain a contradiction

$$|V| > 2 \cdot |B_x(\delta(\Gamma)/4)| \geq 2 \cdot \frac{|V|}{2} \geq |V|.$$

Now assume that $\rho_{1/2}(\Gamma) < \frac{1}{4}$. Then there exists a set $A \subseteq V$ such that

$$|A| \geq \frac{|V|}{2} \quad \text{and} \quad \delta(A) < \frac{\delta(\Gamma)}{4}.$$

Choose any point $x_0 \in A$. Then $A \subseteq B_{x_0}(\delta(\Gamma)/4)$. Thus

$$|A| \leq |B_{x_0}(\delta(\Gamma)/4)| < \frac{|V|}{2}$$

and we again get a contradiction. □

Corollary 7. *The family of all vertex transitive connected graphs on at least three vertices has 1/4-uniform volume distribution.*

We recall an inequality due to Chung [11], who showed that for a k -regular graph

$$\delta(\Gamma) \leq \left\lceil \frac{\log(|V| - 1)}{\log k/\alpha} \right\rceil, \quad (8)$$

where $\alpha = |\alpha_2|$ for the eigenvalues of the adjacency matrix $\alpha_1, \alpha_2 \dots$ satisfying $|\alpha_1| \geq |\alpha_2| \geq \dots \geq \alpha_n$. Below we will compare some of our results with those which follow from inequality (8). Note that in comparison with inequality (6), Theorem 3 automatically gives better convergence as soon as a family of graphs has distortion better than $O(\log |V_i|)$. Indeed, inequality (6) follows from Theorem 3 and Bourgain's upper bound $C_B^{(2)} \log(|V|)$ on the distortion. A version of inequality (8) in the case of directed graphs, including Cayley graphs of finite groups, was also studied recently in [12]. We direct the reader to [13] for details.

4.2. Finite wreath products

Consider the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}_n$ of cyclic groups of order 2 and n , that is the semidirect product $(\oplus_{i=1}^n \mathbb{Z}_2) \rtimes \mathbb{Z}_n$, where the action of \mathbb{Z}_n on $\oplus_{i=1}^n \mathbb{Z}_2$ is given by a coordinate shift. The natural generating set of the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}_n$ is

$$\{(1, 0, 0, \dots, 0), a\},$$

where a is the generator of \mathbb{Z}_n , and one can easily prove that the diameters of the Cayley graphs with respect to this system of generators grow linearly. Spectra of finite wreath products as above were studied in [26]. It was shown in [3] that the Euclidean distortion in this case is $O(\sqrt{\log n})$. Therefore, by Theorem 3 and uniform volume distribution, we have

Proposition 8. *There exists a constant $C > 0$ such that*

$$\lambda_1^{(2)}(\mathbb{Z}_2 \wr \mathbb{Z}_n) \leq C \left(\frac{\sqrt{\log n}}{n} \right)^2 = C \frac{\log n}{n^2}. \quad (9)$$

In comparison, inequality (8) gives

$$\lambda_1^{(2)}(\mathbb{Z}_2 \wr \mathbb{Z}_n) \leq 1 - (n^2 - 1)^{(-1/n)}.$$

Then one can verify that the estimate (9) is asymptotically stronger.

4.3. The group of intermediate growth

Consider the group G of intermediate growth generated by automorphisms a, b, c, d of order 2 of a binary rooted tree. The description of this group can be found in many places, for instance [17, 18, 21], while the spectral properties of Schreier graphs of G were studied in [4]. Without getting into details let us mention that the shape of the Schreier graphs is shown in Figure 1.

We have in this case $\delta(\Gamma_n) = 2^n - 1$. It is easy to see that the ℓ_p -distortion of Γ_n is bounded and that Γ_n have uniform volume distribution with $\rho_{1/2}(\Gamma_n) \geq 1/2$. Thus we obtain from Theorem 3,

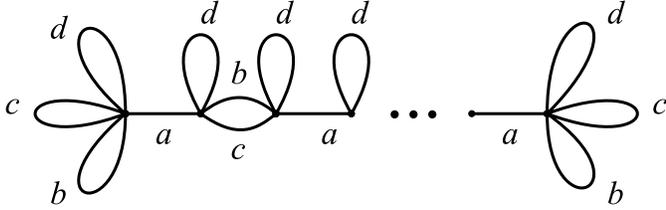


Figure 1: The Schreier graph at level n of the group of intermediate growth has 2^n vertices

Proposition 9. *The p -spectral gap of Γ_n converges exponentially to 0. More precisely, there exists a constant $C > 0$ such that*

$$\lambda_1^{(p)}(\Gamma_n) \leq \frac{C}{2^{np}}.$$

In the case $p = 2$ the spectrum of Γ_n was precisely computed in [4] and gives the asymptotics $C_1 4^{-n} \leq \lambda_1(\Gamma_n) \leq C_2 4^{-n}$. In comparison, inequality (8) gives $\lambda_1^{(2)}(\Gamma_n) \leq C_3 n 2^{-n}$. Here C_1, C_2, C_3 are positive constants.

5. Hanoi Towers group on 3 pegs and its Schreier graphs

The Hanoi Towers game on $k \geq 3$ pegs leads to a famous combinatorial problem [28, 38], which remains unsolved for $k \geq 4$ (although it is solved asymptotically in [38]). The case $k = 3$, despite its easy solution, attracted the attention of many researchers because of its relations to different topics in mathematics, including studies around the Sierpiński gasket [25, 35, 39].

In [23] it was discovered that the problem has an interesting connection to algebra, namely to the theory of self-similar groups, branch groups and iterated monodromy groups.

The natural sequence of graphs related to the game on 3 pegs is a sequence of the so-called Pascal graphs, which is closely related to the sequence of Sierpiński graphs studied in fractal theory [39]. Slight modifications of these graphs also arise as Schreier graphs for the natural action of the associated Hanoi Tower group $H^{(3)}$ on the ternary tree. Their shape is shown in Figure 2. Let us briefly define the corresponding group and the sequence of graphs. We direct the reader to [20, 23, 24, 25] for background on the Hanoi Towers groups.

Let \mathcal{T}_3 denote the 3-regular rooted tree. The set of vertices of \mathcal{T}_3 can be viewed as the set of elements of the free monoid \mathcal{A}^* of finite words over the alphabet $\mathcal{A} = \{0, 1, 2\}$. Given a permutation $\sigma \in \Sigma_3$ in the symmetric group define the automorphism a_σ of \mathcal{T}_3 by the recursive rule

$$a = \sigma(a_0, a_1, a_2),$$

where a_i is the identity automorphism if i is in the support of σ and $a_i = a$ otherwise. The automorphisms $a_{(ij)}$ act on the set of vertices \mathcal{T}_3 by the following

recursive formulas:

$$\begin{aligned} a_{(ij)}(iw) &= jw \\ a_{(ij)}(jw) &= iw \\ a_{(ij)}(xw) &= xa_{(ij)}(w) \end{aligned}$$

for $x \neq i, j$, where $i, j \in \mathcal{A}$ and $w \in \mathcal{A}^*$. Denote $a = a_{(01)}$, $b = a_{(02)}$ and $c = a_{(12)}$. The Hanoi Towers group on 3 pegs is the group $H^3 \subseteq \text{Aut}(\mathcal{T}_3)$ generated by the automorphisms a , b and c . Consider now the subgroup $P_n = \text{st}_{H^3}(1^n) \leq H^3$. The Schreier graph of this subgroup is shown in Figure 3. The spectral theory of Pascal graphs was studied in [24] and [35], from which it follows that $\lambda_1 \leq C5^{-n}$. We will obtain information about the distortion of Schreier graphs (which are graphs of actions on levels) and apply Theorem 3.

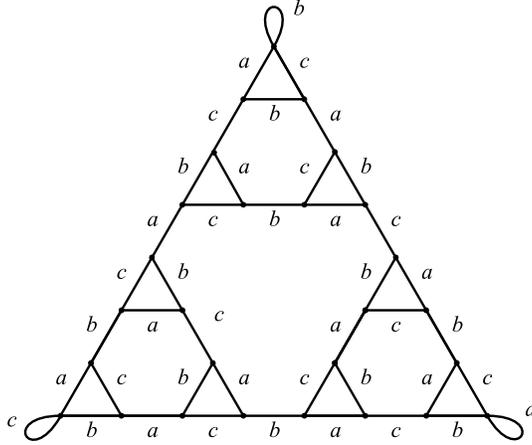


Figure 2: The Schreier graph of $H^{(3)}$ at level 3

Theorem 10. *The graphs Γ_n have uniform volume distribution. More precisely, $\rho_{2/3}(\Gamma_n) \geq 1/2$ for every $n \in \mathbb{N}$.*

Proof. We have $\delta(\Gamma_n) = 2^n - 1$. Denote by S_1 , S_2 and S_3 the three copies of Γ_{n-1} which are naturally embedded in $\Gamma_n = (V, E)$, as in Figure 3. Consider a set $A \subseteq V$ which satisfies $|A| \geq 2/3|V|$. Then we consider two cases.

If $|A| = 2/3|V|$ and $A = S_i \cup S_j$ for some i, j then $\delta(A) = \delta(\Gamma_n)$.

Let A intersect all three S_i , $i = 1, 2, 3$. and let $x_i \in A$, $i = 1, 2, 3$ belong to S_i . Consider two cases. The first one is when the distance between two of the points, say x_2 and x_3 is realized by a geodesic that passes through S_1 . In that case the distance $d(x_3, x_2) \geq \delta(\Gamma_{n-1}) + 2 \geq \delta(\Gamma_n)/2$.

The second case is when the distance between x_i and x_j is not realized by a geodesic through S_k where $i, j, k \in \{1, 2, 3\}$ and are all different. For each

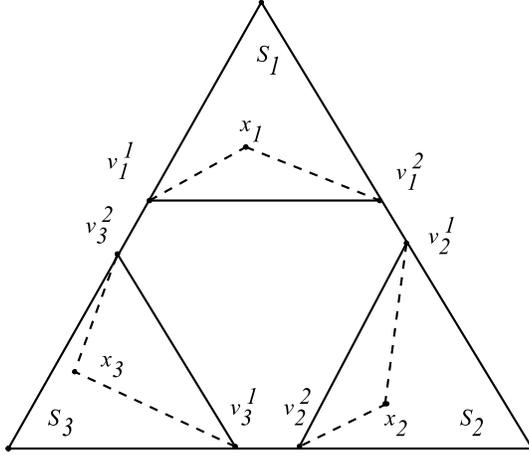


Figure 3: The graph Γ_n

$i = 1, 2, 3$ let v_i^j for $j = 1, 2$ denote the two vertices of the triangle S_i that are not vertices of the large triangle, as shown in Figure 3. Then we have

$$\begin{aligned}
\delta(A) &\geq \max \{d(x_1, x_2), d(x_2, x_3), d(x_3, x_1)\} \\
&\geq \frac{1}{3} (d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1)) \\
&\geq \frac{1}{3} (d(x_1, v_1^2) + d(v_1^1, x_2) + 1 \\
&\quad + d(x_2, v_2^2) + d(v_2^1, x_3) + 1 + d(x_3, v_3^2) + d(v_3^1, x_1) + 1) \\
&\geq \frac{1}{3} (3 + d(v_1^1, v_1^2) + d(v_2^1, v_2^2) + d(v_3^2, v_3^3)) \\
&\geq 1 + \delta(\Gamma_{n-1}) \geq \frac{\delta(\Gamma_n)}{2},
\end{aligned}$$

which finishes the proof. \square

Theorem 11. *For any $p \geq 1$ the ℓ_p -distortion of the sequence $\{\Gamma_n\}$ is uniformly bounded above.*

Before we proceed with the proof we recall the definition of a quasi-isometry, which is a standard notion of metric equivalence in coarse geometry and geometric group theory.

Definition 12. *Let X and Y be metric spaces. A map $f : X \rightarrow Y$ is a quasi-isometry if there exist constants $L \geq 1$ and $K, C \geq 0$ such that*

$$\frac{1}{L}d_X(x, y) - C \leq d_Y(f(x), f(y)) \leq Ld_X(x, y) + C$$

and the image of X is K -dense in Y ; that is, for every $y \in Y$ there exists $x \in X$ such that $d_Y(f(x), y) \leq K$.

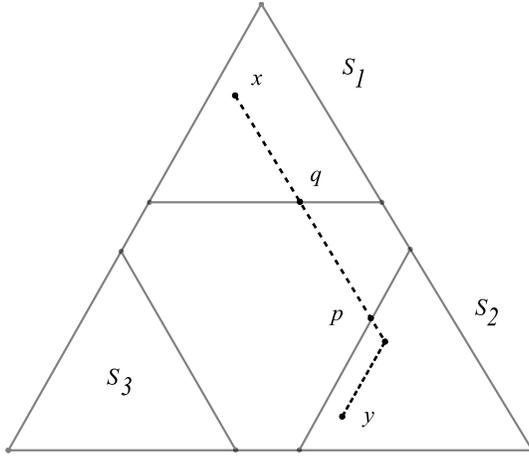


Figure 4: y is to the left of x

Proof of Theorem 11. We first consider the case $p = 2$. Take the triangulation of the plane by equilateral triangles, as in Fig. 6. The set of vertices of this net will be called M . This set can be equipped with the combinatorial metric on the 1-skeleton N of the triangulation. With this metric, N is quasi-isometric to the plane equipped with any p -norm, $p \geq 1$. Indeed, M can be viewed as the Cayley graph of the group \mathbb{Z}^2 with the generating set

$$\{(1, 0), (-1, 0), (0, 1), (0, -1), (-1, 1), (1, -1)\}.$$

Theorem 11 will be proved once we show that the graphs Γ_n are subgraphs of (M, N) with the sets of vertices embedded quasi-isometrically with uniform constants into M .

First, note that Γ_1 embeds into M with distortion less than 3. We will prove by induction on n that Γ_n embeds into M with distortion bounded by 3.

Denote by S_1, S_2, S_3 the three copies of Γ_{n-1} embedded in Γ_n , starting with S_1 being the top one and continuing in the clockwise direction, as in Figure 3. If x, y are vertices in S_i for some $i = 1, 2, 3$ then the claim follows from the inductive assumption. It suffices to prove the estimate for $x \in S_1$ and $y \in S_2$, the other cases being completely analogous. We consider the infinite straight line l passing through x , parallel to that side Γ_n (viewed as a triangle on the plane) which intersects both S_1 and S_2 . The point $y \in S_2$ can be on the left side of l or on the right side of l (see Figures 4 and 5). We consider each case separately.

Case 1: y is to the left of l . Consider the geodesic linking x and y in M which consists of two straight segments: the first segment is contained in l ; the other segment is the segment of a straight line through y parallel to the side of Γ_n which does not intersect S_2 . Let p be the point of intersection of this geodesic with that side of S_2 which is the only side not contained in any side of Γ_n .

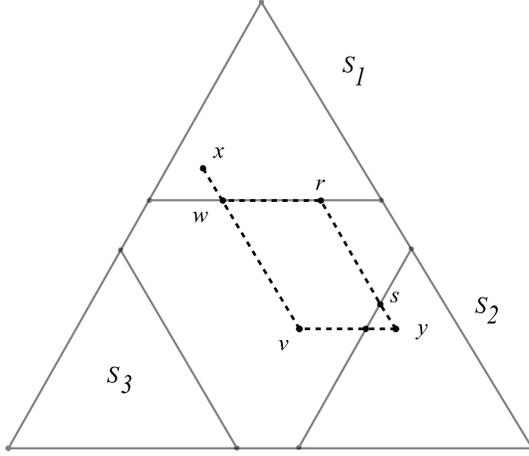


Figure 5: y is to the right of x

Then the geodesic segment $[y, p]$ is completely contained in S_2 and the estimate for distortion for Γ_{n-1} applies. Let q denote the point of intersection of the geodesic with that side of S_1 which is the only side not contained in any side of Γ_n . Then the geodesic segment $[x, q]$ lies completely inside of S_1 and again the estimate for distortion Γ_{n-1} applies.

For the part of the geodesic linking p and q we note that it lies on the line parallel to the side of Γ_n and so we have

$$d_M(p, q) \geq \frac{1}{3}d_{\Gamma_n}(p, q).$$

This estimate follows from the fact that the geodesic linking p and q in M can be viewed as the side of a equilateral triangle of side length $d_M(p, q)$. Then $d_{\Gamma_n}(p, q) \leq 2d_M(p, q) + 1 \leq 3d_M(p, q)$. Summarizing, we have

$$\begin{aligned} d_M(x, y) &= d_M(x, p) + d_M(p, q) + d_M(q, y) \\ &\geq \frac{1}{3}d_{\Gamma_n}(x, q) + \frac{1}{3}d_{\Gamma_n}(q, p) + \frac{1}{3}d_{\Gamma_n}(p, y) \\ &\geq \frac{1}{3}d_{\Gamma_n}(x, y). \end{aligned}$$

Case 2: y is to the right of l . Consider the geodesic γ linking x and y , consisting of a segment of l and a segment of a horizontal straight line between y and the intersection with l . Let v denote the point of intersection of these two lines. This leads to two subcases:

a) If v lies inside of S_2 we use arguments as above.

b) If v lies outside of S_2 we replace the geodesic γ by another geodesic as follows. We take the point of intersection w of the geodesic γ with the boundary of S_1 .

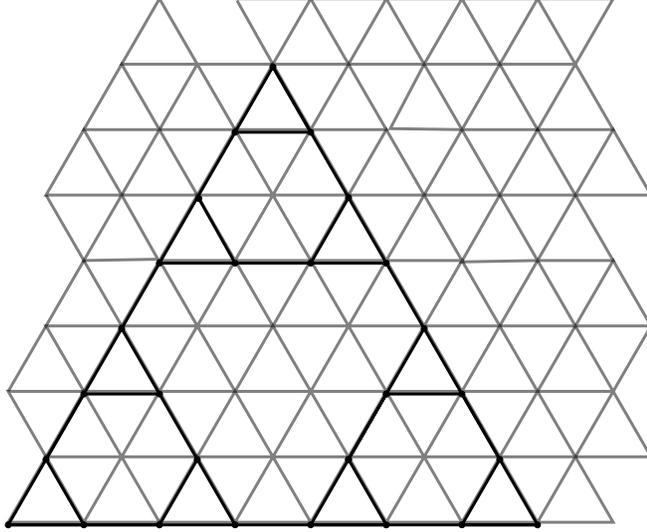


Figure 6: Embedding Γ_n into \mathbb{R}^2 .

The points w and y can now be linked by geodesic γ' which consists of two straight line segments: a segment of the side of S_1 containing w and a segment of a straight line through y parallel to the side of Γ_n intersecting both S_1 and S_2 . Our new geodesic is the segment $d[x, w] \cup \gamma'$, see Figure 5.

Consider now the new geodesic linking x and y to be oriented in the direction from x to y . Let r denote the last point of intersection of the geodesic with S_1 and let s denote the first point of intersection of the geodesic with S_2 . Then again, the segment linking r and s is on a line parallel to one side of Γ_n while the segments from x to r and from s to y are completely contained in S_1 and S_2 respectively. Thus we have an estimate:

$$\begin{aligned}
 d_M(x, y) &= d_M(x, r) + d_M(r, s) + d_M(s, y) \\
 &\geq \frac{1}{3}d_{\Gamma_n}(x, r) + \frac{1}{3}d_{\Gamma_n}(r, s) + \frac{1}{3}d_{\Gamma_n}(s, y) \\
 &\geq \frac{1}{3}d_{\Gamma_n}(x, y).
 \end{aligned}$$

This proves that the ℓ_p -distortion of Γ_n is bounded. □

Applying the above facts together with our inequality we obtain

Theorem 13. *The p -spectral gap of the graphs Γ_n converges exponentially to zero for $1 \leq p < \infty$. More precisely, there exist a constant $C > 0$ (depending on p) such that for every $n \in \mathbb{N}$ we have*

$$\lambda_1^{(p)}(\Gamma_n) \leq C2^{-np}.$$

Thus for $p = 2$ we have $\lambda_1 \leq C4^{-n}$. The exponential convergence in the case $p = 2$ follows from [25]

Remark 14. The Sierpinski graphs [39] are very similar to the graphs Γ_n above, the difference being the three copies of Γ_{n-1} in Γ_n are linked at the vertices, without the connecting edges. The techniques used in this section give conclusions similar as above for the Sierpinski graphs.

6. Remaining questions

There are some problems that we consider interesting in this context. The Basilica group was introduced in [27] and its Schreier graphs were studied in [8, 5, 14]. These graphs can be constructed by a replacement algorithm and have a tree-like structure with cycles of increasing diameters.

Problem 15. *What is the distortion of Schreier graphs of the Basilica group?*

We conjecture that the distortion of these Schreier graphs is unbounded. The spectra of Laplacians on the (slightly modified) Schreier graphs of the Basilica group were computed in [36], while for Schreier graphs associated to Hanoi Tower groups $H^{(k)}$, $k \geq 4$, the spectra are not known. In the latter case it is known that the diameters $\delta(\Gamma_n)$ grow asymptotically as $\exp(n^{1/(k-2)})$ and Chung's inequality gives

$$\lambda_1^{(2)}(\Gamma_n) \leq Cn \exp(-n^{1/(k-2)}).$$

The Hanoi Tower group we considered corresponds to the case $k = 3$ pegs. The question of computing the spectrum of Schreier graphs associated to Hanoi Tower groups on $k \geq 4$ pegs was posed in [24]. Here we state

Problem 16. *What is the asymptotic behavior of $\lambda_1^{(2)}(\Gamma_n)$ for the Schreier graphs associated to the Hanoi Tower groups $H^{(k)}$ for $k \geq 4$?*

The possibility of applying Theorem 3 to the above problem also yields

Problem 17. *What is the distortion of the family of Schreier graphs $\{H_n^{(k)}\}_{n \in \mathbb{N}}$ when $k \geq 4$?*

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