

AMENABLE ACTIONS, INVARIANT MEANS AND BOUNDED COHOMOLOGY

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ABSTRACT. We show that amenability of an action of a discrete group on a compact space X is equivalent to vanishing of bounded cohomology for a class of Banach G -modules associated to the action, that can be viewed as analogs of continuous bundles of dual modules over the G -space X . In the case when the compact space is a point our result reduces to a classic theorem of B.E. Johnson, characterising amenability of groups. In the case when the compact space is the Stone-Ćech compactification of the group we obtain a cohomological characterisation of exactness, or equivalently, Yu's property A for the group, answering a question of Higson.

1. INTRODUCTION

An invariant mean on a countable discrete group G is a positive linear functional on $\ell^\infty(G)$ which is normalised by the requirement that it pairs with the constant function 1 to give 1, and which is fixed by the natural action of G on the space $\ell^\infty(G)^*$. A group is said to be amenable if it admits an invariant mean. The notion of an amenable action of a group on a topological space generalises the concept of amenability and arises naturally in many areas of mathematics. For example, a group acts amenably on a point if and only if it is amenable, while every hyperbolic group acts amenably on its Gromov boundary.

In this paper we introduce the notion of an invariant mean for a topological action and prove that the existence of such a mean characterises amenability of the action. Moreover, we use the existence of the mean to prove vanishing of bounded cohomology of G with coefficients in a suitable class of Banach G modules, and conversely we prove that vanishing of these cohomology groups characterises amenability of the action. This generalises the results of Johnson [9] on bounded cohomology for amenable groups.

Another generalisation of amenability, this time for metric spaces, was given by Yu [14] with the definition of property A. Higson and Roe [10] proved a remarkable result that unifies the two approaches: A finitely generated discrete group G (regarded as a metric space) has Yu's property A if and only if the action of G on its Stone-Ćech compactification βG is topologically amenable, and this is true if and only if G acts amenably on some compact space. Guentner and Kaminker [7] and Ozawa proved [13] that property A and exactness are equivalent for countable discrete groups equipped with a proper left-invariant metric. To generalise the concept of invariant mean to the context of a topological action, we introduce a Banach G -module $W_0(G, X)$ which is an

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analogue of $\ell^1(G)$, encoding both the group and the space on which it acts. Taking the dual and double dual of this space we obtain analogues of $\ell^\infty(G)$ and $\ell^\infty(G)^*$. A mean for the action is an element $\mu \in W_0(G, X)^{**}$ satisfying the normalisation condition $\mu(\pi) = 1$, where the element π is a summation operator, corresponding to the pairing of $\ell^1(G)$ with the constant function 1 in $\ell^\infty(G)$. A mean μ is said to be invariant if $\mu(g \cdot \varphi) = \mu(\varphi)$ for every $\varphi \in W_0(G, X)^*$, (Definition 2).

With these notions in place we give the a very natural characterisation of amenable actions in terms of existence of invariant means and turn to the question of a cohomological characterisation of amenable actions. Given an action of a countable discrete group G on a compact space X by homeomorphisms we introduce a submodule $N_0(G, X)$ of $W_0(G, X)$ associated to the action and which is analogous to the submodule $\ell_0^1(G)$ of $\ell^1(G)$ consisting of all functions of sum 0. Indeed when X is a point these modules coincide. We also define a cohomology class $[J]$, called the Johnson class of the action, which lives in the first bounded cohomology group of G with coefficients in a certain module, denoted $N_0(G, X)^{**}$ (see Section 2 for definitions). Our main result is the following.

Theorem 1. *Let G be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space X . Then the following are equivalent*

- (1) *The action of G on X is topologically amenable.*
- (2) *There exists an invariant mean for the action.*
- (3) *The class $[J] \in H_b^1(G, N_0(G, X)^{**})$ is trivial.*
- (4) *$H_b^p(G, \mathcal{E}^*) = 0$ for $p \geq 1$ and every ℓ^1 -geometric G - $C(X)$ module \mathcal{E} .*

Subsequent to our result Monod produced a similar theorem which can be found in [12].

The definition of ℓ^1 -geometric G - $C(X)$ module is given in Section 2. Choosing the correct class of modules as coefficients is one of the key new elements of the proof. The main idea is that the duals of these modules have a structure that makes them similar to a continuous bundle of dual modules over the space X . This structure is obtained by two new ingredients. The first one is an additional structure of a module over $C(X)$, similarly as in the classical Serre-Swan theorem characterizing finite dimensional bundles. The second is a norm condition, defined using the representation of $C(X)$, which mimics the behavior of the norm in the space $C(X, E)$ of continuous functions from X to a normed space E .

When X is a point, our theorem reduces to Johnson's celebrated characterisation of amenability [9]. As a corollary we also obtain a cohomological characterisation of exactness for discrete groups, which answers a question of Higson, and which follows from our main result when X is the Stone-Ćech compactification βG of the group G . In this case, $C(\beta G)$ can be identified with $\ell^\infty(G)$, and we obtain the following.

Corollary. *Let G be a countable discrete group. Then the following are equivalent.*

- (1) *The group G is exact;*

- (2) *The Johnson class $[J] \in H_b^1(G, N_0(G, \beta G)^{**})$ is trivial;*
- (3) *$H_b^p(G, \mathcal{E}^*) = 0$ for $p \geq 1$ and every ℓ^1 -geometric G - $\ell^\infty(G)$ -module \mathcal{E} .*

It should be pointed out that bounded cohomology groups are difficult to compute, while our results apply to many examples of amenable actions. For instance, as mentioned earlier, hyperbolic groups act amenably on their boundary [1]. Another example is furnished by mapping class groups of surfaces, which act amenably on the space of complete geodesic laminations of the surface [8]. Finally, the case of exact groups and actions on the Stone-Ćech compactification was discussed above. In all these case we obtain vanishing of bounded cohomology for a large class of natural G -modules.

This paper builds on the cohomological characterisation of property A introduced in [3] and on the study of cohomological properties of exactness in [5]. The ideas were developed during the visit to Southampton by Nowak in January 2010.

2. GEOMETRIC BANACH MODULES

Let $C(X)$ denote the space of real-valued continuous functions on X . For a function $f : G \rightarrow C(X)$ we shall denote by f_g the continuous function on X obtained by evaluating f at $g \in G$. We define the sup- ℓ^1 norm of f to be

$$\|f\|_{\infty,1} = \sup_{x \in X} \sum_{g \in G} |f_g(x)|,$$

and denote by V the Banach space of all functions on G with values in $C(X)$ that have finite norm. We introduce a Banach G -module associated to the action.

Definition 2. *Let $W_{00}(G, X)$ be the subspace of V consisting of all functions $f : G \rightarrow C(X)$ which have finite support and such that for some $c \in \mathbb{R}$, depending on f , $\sum_{g \in G} f_g = c1_X$, where 1_X denotes the constant function 1 on X . The closure of this space in the sup- ℓ^1 -norm will be denoted $W_0(G, X)$.*

Let $\pi : W_{00}(G, X) \rightarrow \mathbb{R}$ be defined by $\sum_{g \in G} f_g = \pi(f)1_X$. The map π is continuous with respect to the sup- ℓ^1 norm and so extends to the closure $W_0(G, X)$; we denote its kernel by $N_0(G, X)$.

In the case of $X = \beta G$ and $C(\beta G) = \ell^\infty(G)$ the space $W_0(G, \beta G)$ was introduced in [6]. For every $g \in G$ we define the function $\delta_g \in W_{00}(G, X)$ by $\delta_g(h) = 1_X$ when $g = h$, and zero otherwise.

The G -action on X gives an isometric action of G on $C(X)$ in the usual way: for $g \in G$ and $f \in C(X)$, we have $(g \cdot f)(x) = f(g^{-1}x)$. The group G also acts isometrically on the space V in a natural way: for $g, h \in G, f \in V, x \in X$, we have $(gf)_h(x) = f_{g^{-1}h}(g^{-1}x) = (g \cdot f_{g^{-1}h})(x)$.

Since the summation map π is G -equivariant (we assume that the action of G on \mathbb{R} is trivial) the action of G restricts to $W_{00}(G, X)$ and so by continuity it restricts to $W_0(G, X)$. We obtain a short exact sequence of G -vector spaces:

$$0 \rightarrow N_0(G, X) \rightarrow W_0(G, X) \xrightarrow{\pi} \mathbb{R} \rightarrow 0.$$

Definition 3. Let \mathcal{E} be a Banach space. We say that \mathcal{E} is a $C(X)$ -module if it is equipped with a contractive unital representation of the Banach algebra $C(X)$.

If X is a G -space then a $C(X)$ -module \mathcal{E} is said to be a G - $C(X)$ -module if the group G acts on \mathcal{E} by isometries and the representation of $C(X)$ is G -equivariant.

Note that the fact that we will only ever consider unital representations of $C(X)$ means that there is no confusion between multiplying by a scalar or by the corresponding constant function. For instance, for $f \in W_0(G, X)$ multiplication by $\pi(f)$ agrees with multiplication by $\pi(f)1_X$.

Example 4. The space V is a G - $C(X)$ -module. Indeed, for every $f \in V$ and $t \in C(X)$ we define $tf \in V$ by $(tf)_g(x) = t(x)f_g(x)$, for all $g \in G$. This action is well-defined as $\|tf\|_{\infty,1} \leq \|t\|_{\infty}\|f\|_{\infty,1}$; this also implies that the representation of $C(X)$ on V is contractive. As remarked above, the group G acts isometrically on V . The representation of $C(X)$ is clearly unital and also equivariant, since for every $g \in G$, $f \in V$ and $t \in C(X)$

$$(g(tf))_h(x) = (tf)_{g^{-1}h}(g^{-1}x) = t(g^{-1}x)f_{g^{-1}h}(g^{-1}x) = (g \cdot t)(x)(gf)_h(x)$$

Thus we have $g(tf) = (g \cdot t)(gf)$.

The equivariance of the summation map π implies that both $W_0(G, X)$ and $N_0(G, X)$ are G -invariant subspaces of V . Note however, that $W_0(G, X)$ is not invariant under the action of $C(X)$ defined above, as for $f \in W_0(G, X)$ and $t \in C(X)$ we have

$$\sum_{g \in G} (tf)_g(x) = \sum_{g \in G} t(x)f_g(x) = t(x) \sum_{g \in G} f_g(x) = ct(x).$$

However, the same calculation shows that the subspace $N_{00}(G, X)$ is invariant under the action of $C(X)$, and so is a G - $C(X)$ -module, and hence so is its closure $N_0(G, X)$.

Let \mathcal{E} be a G - $C(X)$ -module, let \mathcal{E}^* be the Banach dual of \mathcal{E} and let $\langle -, - \rangle$ be the pairing between the two spaces. The induced actions of G and $C(X)$ on \mathcal{E}^* are defined as follows. For $\alpha \in \mathcal{E}^*$, $g \in G$, $f \in C(X)$, and $v \in \mathcal{E}$ we let

$$\langle g\alpha, v \rangle = \langle \alpha, g^{-1}v \rangle, \quad \langle f\alpha, v \rangle = \langle \alpha, fv \rangle.$$

Note that the action of $C(X)$ is well-defined since $C(X)$ is commutative. It is easy to check the following.

Lemma 5. *If \mathcal{E} is a G - $C(X)$ -module, then so is \mathcal{E}^* .*

We will now introduce a geometric condition on Banach modules which will play the role of an orthogonality condition. To motivate the definition that follows, let us note that if f_1 and f_2 are functions with disjoint supports on a space X then (assuming that the relevant norms are finite) the sup-norm satisfies the identity $\|f_1 + f_2\|_{\infty} = \sup\{\|f_1\|_{\infty}, \|f_2\|_{\infty}\}$, while for the ℓ^1 -norm we have $\|f_1 + f_2\|_{\ell^1} = \|f_1\|_{\ell^1} + \|f_2\|_{\ell^1}$.

Definition 6. Let \mathcal{E} be a Banach space and a $C(X)$ -module. We say that v_1 and v_2 in \mathcal{E} are disjointly supported if there exist $f_1, f_2 \in C(X)$ with disjoint supports such that $f_1 v_1 = v_1$ and $f_2 v_2 = v_2$.

We say that the module \mathcal{E} is ℓ^∞ -geometric if, whenever v_1 and v_2 have disjoint supports, $\|v_1 + v_2\| = \sup\{\|v_1\|, \|v_2\|\}$.

We say that the module \mathcal{E} is ℓ^1 -geometric if for every two disjointly supported v_1 and v_2 in \mathcal{E} $\|v_1 + v_2\| = \|v_1\| + \|v_2\|$.

If v_1 and v_2 are disjointly supported elements of \mathcal{E} and f_1 and f_2 are as in the definition, then $f_1 v_2 = f_1 f_2 v_2 = 0$, and similarly $f_2 v_1 = 0$.

Note also that the functions f_1 and f_2 can be chosen to be of norm one in the supremum norm on $C(X)$. To see this, note that Tietze's extension theorem allows one to construct continuous functions f'_1, f'_2 on X which are of norm one, have disjoint supports and such that f'_i takes the value 1 on $\text{Supp } f_i$. Then $f'_i \phi_i = (f'_i f_i) \phi_i = f_i \phi_i = \phi_i$. Now replace f_i with f'_i .

Finally, if $f_1, f_2 \in C(X)$ have disjoint supports then, again by Tietze's extension theorem, $f_1 v_1$ and $f_2 v_2$ are disjointly supported for all $v_1, v_2 \in \mathcal{E}$.

Examples of ℓ^∞ -geometric G - $C(X)$ modules include the modules $C(X, E)$, the space of continuous functions from a G -space X to a G -module E and any of its submodules closed under both G - and $C(X)$ - module structures. Examples of ℓ^1 -geometric modules can be then produced using the following fact.

Lemma 7. If \mathcal{E} is an ℓ^1 -geometric module then \mathcal{E}^* is ℓ^∞ -geometric.

If \mathcal{E} is an ℓ^∞ -geometric module then \mathcal{E}^* is ℓ^1 -geometric.

Proof. Let us assume that $\phi_1, \phi_2 \in \mathcal{E}^*$ are disjointly supported and let $f_1, f_2 \in C(X)$ be as in Definition 6, chosen to be of norm 1.

If \mathcal{E} is ℓ^1 -geometric, then for every vector $v \in \mathcal{E}$, $\|f_1 v\| + \|f_2 v\| = \|(f_1 + f_2)v\| \leq \|v\|$. Furthermore,

$$\begin{aligned} \|\phi_1 + \phi_2\| &= \sup_{\|v\|=1} |\langle \phi_1 + \phi_2, v \rangle| = \sup_{\|v\|=1} |\langle f_1 \phi_1, v \rangle + \langle f_2 \phi_2, v \rangle| \\ &= \sup_{\|v\|=1} |\langle \phi_1, f_1 v \rangle + \langle \phi_2, f_2 v \rangle| \\ &\leq \sup_{\|v\|=1} (\|\phi_1\| \|f_1 v\| + \|\phi_2\| \|f_2 v\|) \\ &\leq \sup\{\|\phi_1\|, \|\phi_2\|\} \sup_{\|v\|=1} (\|f_1 v\| + \|f_2 v\|) \\ &\leq \sup\{\|\phi_1\|, \|\phi_2\|\} \end{aligned}$$

Since $f_1 \phi_2 = 0$ we have that

$$\|\phi_1\| = \|f_1(\phi_1 + \phi_2)\| \leq \|f_1\| \|\phi_1 + \phi_2\| = \|\phi_1 + \phi_2\|.$$

Similarly, we have $\|\phi_2\| \leq \|\phi_1 + \phi_2\|$, and the two estimates together ensure that $\|\phi_1 + \phi_2\| = \sup\{\|\phi_1\|, \|\phi_2\|\}$ as required.

For the second statement, let us assume that \mathcal{E} is ℓ^∞ -geometric and that $\phi_1, \phi_2 \in \mathcal{E}^*$ are disjointly supported. Then

$$\begin{aligned} \|\phi_1\| + \|\phi_2\| &= \sup_{\|v_1\|, \|v_2\|=1} \langle \phi_1, v_1 \rangle + \langle \phi_2, v_2 \rangle \\ &= \sup_{\|v_1\|, \|v_2\|=1} \langle \phi_1, f_1 v_1 \rangle + \langle \phi_2, f_2 v_2 \rangle \\ &= \sup_{\|v_1\|, \|v_2\|=1} \langle \phi_1 + \phi_2, f_1 v_1 + f_2 v_2 \rangle \\ &\leq \sup_{\|v_1\|, \|v_2\|=1} \|\phi_1 + \phi_2\| \|f_1 v_1 + f_2 v_2\| \\ &\leq \|\phi_1 + \phi_2\| \leq \|\phi_1\| + \|\phi_2\|. \end{aligned}$$

where the last inequality is just the triangle inequality, so the inequalities are equalities throughout and $\|\phi_1\| + \|\phi_2\| = \|\phi_1 + \phi_2\|$ as required. \square

We have already established that $N_0(G, X)$ is a G - $C(X)$ -module. Let ϕ^1 and ϕ^2 be disjointly supported elements of $N_0(G, X)$; this means that there exist disjointly supported functions f_1 and f_2 in $C(X)$ such that $\phi^i = f_i \phi^i$ for $i = 1, 2$. Then

$$\|\phi^1 + \phi^2\|_{\infty, 1} = \|f_1 \phi^1 + f_2 \phi^2\| = \sup_{x \in X} \sum_{g \in G} |f_1(x) \phi_g^1(x) + f_2(x) \phi_g^2(x)|$$

We note that the two terms on the right are disjointly supported functions on X and so

$$\|\phi^1 + \phi^2\|_{\infty, 1} = \sup_{x \in X} \left(\sum_{g \in G} |f_1(x) \phi_g^1(x)| + \sum_{g \in G} |f_2(x) \phi_g^2(x)| \right) = \sup(\|\phi^1\|_{\infty, 1}, \|\phi^2\|_{\infty, 1}).$$

Thus we obtain

Lemma 8. *The module $N_0(G, X)$ is ℓ^∞ -geometric. Hence the dual $N_0(G, X)^*$ is ℓ^1 -geometric and the double dual $N_0(G, X)^{**}$ is ℓ^∞ -geometric.*

We now assume that \mathcal{E} is an ℓ^1 -geometric $C(X)$ -module, so that its dual \mathcal{E}^* is ℓ^∞ -geometric.

Lemma 9. *Let $f_1, f_2 \in C(X)$ be non-negative functions such that $f_1 + f_2 \leq 1_X$. Then for every $\phi_1, \phi_2 \in \mathcal{E}^*$*

$$\|f_1 \phi_1 + f_2 \phi_2\| \leq \sup\{\|\phi_1\|, \|\phi_2\|\}.$$

Proof. Let $M \in \mathbb{N}$ and $\varepsilon = 1/M$. For $i = 1, 2$ define $f_{i,0} = \min\{f_i, \varepsilon\}$, $f_{i,1} = \min\{f_i - f_{i,0}, \varepsilon\}$, $f_{i,2} = \min\{f_i - f_{i,0} - f_{i,1}, \varepsilon\}$, and so on, to $f_{i,M-1}$.

Then $f_{i,j}(x) = 0$ iff $f_i(x) \leq j\varepsilon$, so $f_{i,j} > 0$ iff $f_i(x) > j\varepsilon$ which implies that $\text{Supp } f_{i,j} \subseteq f_i^{-1}([j\varepsilon, \infty))$. So for $j \geq 2$, $\text{Supp}(f_{1,j}) \subseteq f_1^{-1}([j\varepsilon, \infty))$ and $\text{Supp } f_{2,M+1-j} \subseteq f_2^{-1}([(M+1-j)\varepsilon, \infty))$.

If $x \in \text{Supp}(f_{1,j}) \cap \text{Supp}(f_{2,M+1-j})$ then $1 \geq f_1(x) + f_2(x) \geq j\varepsilon + (M+1-j)\varepsilon = 1 + \varepsilon$, so the two supports $\text{Supp}(f_{1,j})$, $\text{Supp}(f_{2,M+1-j})$ are disjoint.

We have that

$$f_1 = f_{1,0} + f_{1,1} + \sum_{j=2}^{M-1} f_{1,j}$$

$$f_2 = f_{2,0} + f_{2,1} + \sum_{j=2}^{M-1} f_{2,M+1-j}.$$

So using the fact that $\|f_{1,j}\phi_1 + f_{2,M+1-j}\phi_2\| \leq \sup\{\|f_{1,j}\phi_1\|, \|f_{2,M+1-j}\phi_2\|\} \leq \varepsilon \sup_i \|\phi_i\|$ we have the following estimate:

$$\begin{aligned} \|f_1\phi_1 + f_2\phi_2\| &\leq \|(f_{1,0} + f_{1,1})\phi_1\| + \|(f_{2,0} + f_{2,1})\phi_2\| + \sum_{j=2}^M \|f_{1,j}\phi_1 + f_{2,M+1-j}\phi_2\| \\ &\leq 4\varepsilon \sup_j \|\phi_i\| + \sum_{j=2}^{M-1} \varepsilon \sup_i \|\phi_i\| \\ &= (4\varepsilon + (M-2)\varepsilon) \sup_i \|\phi_i\| \\ &= (1 + 2\varepsilon) \sup_i \|\phi_i\|. \end{aligned}$$

□

Lemma 10. Let $f_1, \dots, f_N \in C(X)$, $f_i \geq 0$, $\sum_{i=1}^N f_i \leq 1_X$, $\phi_1, \dots, \phi_N \in \mathcal{E}^*$.

Then $\|\sum_i f_i \phi_i\| \leq \sup_{1, \dots, N} \|\phi_i\|$.

Proof. We proceed by induction. Assume that the statement is true for some N . Then let $f_0, f_1, \dots, f_N \in C(X)$, $f_i \geq 0$, $\sum_{i=1}^N f_i \leq 1_X$, and let $\phi_0, \phi_1, \dots, \phi_N \in \mathcal{E}^*$.

Let $f'_1 = f_0 + f_1$ and leave the other functions unchanged. For $\delta > 0$ let

$$\phi'_{1,\delta} = \frac{1}{f_0 + f_1 + \delta} (f_0 \phi_0 + f_1 \phi_1).$$

Since we clearly have

$$\frac{f_0}{f_0 + f_1 + \delta} + \frac{f_1}{f_0 + f_1 + \delta} \leq 1_X$$

by the previous lemma we have that $\|\phi'_{1,\delta}\| \leq \sup\{\|\phi_0\|, \|\phi_1\|\}$, and so by induction

$$\|f'_1 \phi'_{1,\delta} + f_2 \phi_2 + \dots + f_N \phi_N\| \leq \sup\{\|\phi'_{1,\delta}\|, \|\phi_2\|, \dots, \|\phi_N\|\} \leq \sup_{i=0, \dots, N} \|\phi_i\|.$$

Consider now

$$f'_1 \phi'_{1,\delta} = \frac{(f_0 + f_1)}{f_0 + f_1 + \delta} (f_0 \phi_0 + f_1 \phi_1) = \frac{(f_0 + f_1)f_0}{f_0 + f_1 + \delta} \phi_0 + \frac{(f_0 + f_1)f_1}{f_0 + f_1 + \delta} \phi_1.$$

We note that for $i = 0, 1$

$$f_i - \frac{(f_0 + f_1)f_i}{f_0 + f_1 + \delta} = \frac{\delta f_i}{f_0 + f_1 + \delta} \leq \delta$$

and so $\frac{(f_0 + f_1)f_i}{f_0 + f_1 + \delta}$ converges to f_i uniformly on X , as $\delta \rightarrow 0$, which implies that $f'_1 \phi'_{1,\delta}$ converges to $f_0 \phi_0 + f_1 \phi_1$ in norm, and the lemma follows. \square

Lemma 11. *If $f_1, \dots, f_N \in C(X)$ (we do not assume that $f_i \geq 0$) are such that $\sum_{i=1}^N |f_i| \leq 1_X$ and $\phi_1, \dots, \phi_N \in \mathcal{E}^*$ then*

$$\left\| \sum_{i=1}^N f_i \phi_i \right\| \leq 2 \sup_{i=1, \dots, N} \|\phi_i\|.$$

Proof. If $f_i = f_i^+ - f_i^-$, then $|f_i| = f_i^+ + f_i^-$ and $\sum f_i^+ + \sum f_i^- \leq 1$.

Then by the previous lemma $\|\sum_{i=1}^N f_i^\pm \phi_i\| \leq \sup_{i=1, \dots, N} \|\phi_i\|$ so

$$\left\| \sum f_i^+ \phi_i - \sum f_i^- \phi_i \right\| \leq 2 \sup_{i=1, \dots, N} \|\phi_i\|.$$

\square

3. AMENABLE ACTIONS AND INVARIANT MEANS

In this section we will recall the definition of a topologically amenable action [2] and characterise it in terms of the existence of a certain averaging operator. For our purposes the following definition, adapted from [5, Definition 4.3.1] is convenient.

Definition 12. *The action of G on X is amenable if and only if there exists a sequence of elements $f^n \in W_{00}(G, X)$ such that*

- (1) $f_g^n \geq 0$ in $C(X)$ for every $n \in \mathbb{N}$ and $g \in G$,
- (2) $\pi(f^n) = 1$ for every n ,
- (3) for each $g \in G$ we have $\|f^n - g f^n\|_V \rightarrow 0$.

Note that when X is a point the above conditions reduce to the definition of amenability of G . On the other hand, if $X = \beta G$, the Stone-Ćech compactification of G then amenability of the natural action of G on X is equivalent to Yu's property A by a result of Higson and Roe [10].

Remark 13. In the above definition we may omit condition 1 at no cost, since given a sequence of functions satisfying conditions 2 and 3 we can make them positive by replacing each $f_g^n(x)$ by

$$\frac{|f_g^n(x)|}{\sum_{h \in G} |f_h^n(x)|}.$$

Conditions 1 and 2 are now clear, while condition 3 follows from standard estimates (see e.g. [6, Lemma 4.9]).

The first definition of amenability of a group G given by von Neumann was in terms of the existence of an invariant mean on the group. The following definition gives a version of an invariant mean for an amenable action on a compact space.

Definition 14. *Let G be a countable group acting on a compact space X by homeomorphisms. A mean for the action is an element $\mu \in W_0(G, X)^{**}$ such that $\mu(\pi) = 1$. A mean μ is said to be invariant if $\mu(g\varphi) = \mu(\varphi)$ for every $\varphi \in W_0(G, X)^*$.*

The following theorem will be an important tool in the proof of our main result.

Theorem 15. *Let G be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space X . The action is amenable if and only if there exists an invariant mean for the action.*

Proof. Let G act amenably on X and consider the sequence f^n provided by Definition 12. Each f^n satisfies $\|f^n\| = 1$. We now view the functions f^n as elements of the double dual $W_0(G, X)^{**}$. By the weak-* compactness of the unit ball there is a convergent subnet f^λ , and we define μ to be its weak-* limit. It is then easy to verify that μ is a mean. Since

$$|\langle f^\lambda - gf^\lambda, \varphi \rangle| \leq \|f^\lambda - gf^\lambda\|_V \|\varphi\|$$

and the right hand side tends to 0, we obtain $\mu(\varphi) = \mu(g\varphi)$.

Conversely, by Goldstine's theorem, (see, e.g., [11, Theorem 2.6.26]) as $\mu \in W_0(G, X)^{**}$, μ is the weak-* limit of a bounded net of elements $f^\lambda \in W_0(G, X)$. We note that we can choose f^λ in such a way that $\pi(f^\lambda) = 1$. Indeed, given f^λ with $\pi(f^\lambda) = c_\lambda \rightarrow \mu(\pi) = 1$ we replace each f^λ by

$$f^\lambda + (1 - c_\lambda)\delta_e.$$

Since $(1 - c_\lambda)\delta_e \rightarrow 0$ in norm in $W_0(G, X)$, μ is the weak-* limit of the net $f^\lambda + (1 - c_\lambda)\delta_e$ as required.

Since μ is invariant, we have that for every $g \in G$, $gf^\lambda \rightarrow g\mu = \mu$, so that $gf^\lambda - f^\lambda \rightarrow 0$ in the weak-* topology. However, for every $g \in G$, $gf^\lambda - f^\lambda \in W_0(G, X)$, and so the convergence is in fact in the weak topology on $W_0(G, X)$.

For every λ , we regard the family $(gf^\lambda - f^\lambda)_{g \in G}$ as an element of the product $\prod_{g \in G} W_0(G, X)$, noting that this sequence converges to 0 in the Tychonoff weak topology.

Now $\prod_{g \in G} W_0(G, X)$ is a Fréchet space in the Tychonoff norm topology, so by Mazur's theorem there exists a sequence f^n of convex combinations of f^λ such that $(gf^n - f^n)_{g \in G}$ converges to zero in the Fréchet topology. Thus there exists a sequence f^n of elements of $W_0(G, X)$ such that for every $g \in G$, $\|gf^n - f^n\| \rightarrow 0$ in $W_0(G, X)$.

The result now follows after applying the normalisation procedure explained in Remark 13. \square

4. EQUIVARIANT MEANS ON GEOMETRIC MODULES

Given an invariant mean $\mu \in W_0(G, X)^{**}$ for the action of G on X and an ℓ^1 -geometric G - $C(X)$ module \mathcal{E} , we define a G -equivariant averaging operator $\mu_{\mathcal{E}} : \ell^\infty(G, \mathcal{E}^*) \rightarrow \mathcal{E}^*$ which we will also refer to as an equivariant mean for the action.

To do so, following an idea from [4], we introduce a linear functional $\sigma_{\tau, v}$ on $W_{00}(G, X)$. Given a Banach space \mathcal{E} define $\ell^\infty(G, \mathcal{E})$ to be the space of functions $f : G \rightarrow \mathcal{E}$ such that $\sup_{g \in G} \|f(g)\|_{\mathcal{E}} < \infty$. If G acts on \mathcal{E} then the action of the group G on the space $\ell^\infty(G, \mathcal{E})$ is defined in an analogous way to the action of G on V , using the induced action of G on \mathcal{E} :

$$(g\tau)_h = g(\tau_{g^{-1}h}),$$

for $\tau \in \ell^\infty(G, \mathcal{E})$ and $g \in G$.

Let us assume that \mathcal{E} is an ℓ^1 -geometric G - $C(X)$ module, and let $\tau \in \ell^\infty(G, \mathcal{E}^*)$. Choose a vector $v \in \mathcal{E}$ and define a linear functional $\sigma_{\tau, v} : W_{00}(G, X) \rightarrow \mathbb{R}$ by

$$(1) \quad \sigma_{\tau, v}(f) = \left\langle \sum_{h \in G} f_h \tau_h, v \right\rangle$$

for every $f \in W_{00}(G, X)$. If we now use Lemma 11 together with the support condition required of elements of $W_{00}(G, X)$ then we have the estimate

$$|\sigma_{\tau, v}(f)| \leq \left\| \sum_h f_h \tau_h \right\| \|v\| \leq 2\|f\| \|\tau\| \|v\|.$$

This estimate completes the proof of the following.

Lemma 16. *Let \mathcal{E} be an ℓ^1 -geometric G - $C(X)$ module. For every $\tau \in \ell^\infty(G, \mathcal{E}^*)$ and every $v \in \mathcal{E}$ the linear functional $\sigma_{\tau, v}$ on $W_{00}(G, X)$ is continuous and so it extends to a continuous linear functional on $W_0(G, X)$.*

Lemma 17. *The map $\ell^\infty(G, \mathcal{E}^*) \times \mathcal{E} \rightarrow W_0(G, X)^*$ defined by $(\tau, v) \mapsto \sigma_{\tau, v}$ is G -equivariant.*

Proof.

$$\begin{aligned} \sigma_{g\tau, gv}(f) &= \left\langle \sum_h f_h g(\tau_{g^{-1}h}), gv \right\rangle = \left\langle g \sum_h (g^{-1} \cdot f_h) \tau_{g^{-1}h}, gv \right\rangle \\ &= \left\langle \sum_h (g^{-1} \cdot f_h) \tau_{g^{-1}h}, v \right\rangle = \left\langle \sum_h (g^{-1}f)_{g^{-1}h} \tau_{g^{-1}h}, v \right\rangle \\ &= \sigma_{\tau, v}(g^{-1}f) = (g\sigma_{\tau, v})(f). \end{aligned}$$

□

Definition 18. *Let \mathcal{E} be an ℓ^1 -geometric G - $C(X)$ module, and let $\mu \in W_0(G, X)^{**}$ be an invariant mean for the action. We define $\mu_{\mathcal{E}} : \ell^\infty(G, \mathcal{E}^*) \rightarrow \mathcal{E}^*$ by*

$$\langle \mu_{\mathcal{E}}(\tau), v \rangle = \langle \mu, \sigma_{\tau, v} \rangle,$$

for every $\tau \in \ell^\infty(G, \mathcal{E}^*)$, and $v \in \mathcal{E}$.

Lemma 19. *Let \mathcal{E} be an ℓ^1 -geometric G - $C(X)$ module, and let $\mu \in W_0(G, X)^{**}$ be an invariant mean for the action.*

- (1) *The map $\mu_{\mathcal{E}}$ defined above is G -equivariant.*
- (2) *If $\tau \in \ell^\infty(G, \mathcal{E}^*)$ is constant then $\mu_{\mathcal{E}}(\tau) = \tau_e$.*

Proof.

$$\begin{aligned} \langle \mu_{\mathcal{E}}(g\tau), v \rangle &= \mu(\sigma_{g\tau, v}) = \mu(g \cdot \sigma_{\tau, g^{-1}v}) = \mu(\sigma_{\tau, g^{-1}v}) \\ &= \langle \mu_{\mathcal{E}}(\tau), g^{-1}v \rangle = \langle g \cdot (\mu_{\mathcal{E}}(\tau)), v \rangle. \end{aligned}$$

If τ is constant then

$$\begin{aligned} \sigma_{\tau, v}(f) &= \left\langle \sum_h f_h \tau_h, v \right\rangle = \left\langle \left(\sum_h f_h \right) \tau_e, v \right\rangle \\ &= \langle (\pi(f)1_X) \tau_e, v \rangle = \langle \pi(f) \tau_e, v \rangle = \langle \tau_e, v \rangle \pi(f). \end{aligned}$$

So $\sigma_{\tau, v} = \langle \tau_e, v \rangle \pi$ and

$$\langle \mu_{\mathcal{E}}(\tau), v \rangle = \mu(\sigma_{\tau, v}) = \mu(\langle \tau_e, v \rangle \pi) = \langle \tau_e, v \rangle,$$

hence $\mu_{\mathcal{E}}(\tau) = \tau_e$. □

5. AMENABLE ACTIONS AND BOUNDED COHOMOLOGY

Let \mathcal{E} be a Banach space equipped with an isometric action by G . Then we consider a cochain complex $C_b^m(G, \mathcal{E}^*)$ which in degree m consists of G -equivariant bounded cochains $\phi : G^{m+1} \rightarrow \mathcal{E}^*$ with values in the Banach dual \mathcal{E}^* of \mathcal{E} which is equipped with the natural differential d as in the homogeneous bar resolution. Bounded cohomology with coefficients in \mathcal{E}^* will be denoted by $H_b^*(G, \mathcal{E}^*)$.

Definition 20. *Let G be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space X . The function*

$$J(g_0, g_1) = \delta_{g_1} - \delta_{g_0}$$

*is a bounded cochain of degree 1 with values in $N_{00}(G, X)$, and in fact it is a bounded cocycle and so represents a class in $H_b^1(G, N_0(G, X))^{**}$, where we regard $N_{00}(G, X)$ as a subspace of $N_0(G, X)^{**}$. We call $[J]$ the Johnson class of the action.*

We are now in the position to prove our main result, which we restate.

Theorem 1. *Let G be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space X . Then the following are equivalent*

- (1) *The action of G on X is topologically amenable.*
- (2) *There exists an invariant mean for the action.*

(3) The class $[J] \in H_b^1(G, N_0(G, X)^{**})$ is trivial.

(4) $H_b^p(G, \mathcal{E}^*) = 0$ for $p \geq 1$ and every ℓ^1 -geometric G - $C(X)$ module \mathcal{E} .

Proof. The equivalence of (1) and (2) was proved in Theorem 15. We now show that (2) is equivalent to (3). The short exact sequence of G -modules

$$0 \rightarrow N_0(G, X) \rightarrow W_0(G, X) \xrightarrow{\pi} \mathbb{R} \rightarrow 0$$

leads, by taking double duals, to the short exact sequence

$$0 \rightarrow N_0(G, X)^{**} \rightarrow W_0(G, X)^{**} \rightarrow \mathbb{R} \rightarrow 0$$

which in turn gives rise to a long exact sequence in bounded cohomology

$$H_b^0(G, N_0(G, X)^{**}) \rightarrow H_b^0(G, W_0(G, X)^{**}) \rightarrow H_b^0(G, \mathbb{R}) \rightarrow H_b^1(G, N_0(G, X)^{**}) \rightarrow \dots$$

The Johnson class $[J]$ is the image of the class $[1] \in H_b^0(G, \mathbb{R})$ under the connecting homomorphism $d : H_b^0(G, \mathbb{R}) \rightarrow H_b^1(G, N_0(G, X)^{**})$, and so $[J] = 0$ if and only if $d[1] = 0$. By exactness of the cohomology sequence, this is equivalent to $[1] \in \text{Im } \pi^{**}$, where $\pi^{**} : H_b^0(G, W_0(G, X)^{**}) \rightarrow H_b^0(G, \mathbb{R})$ is the map on cohomology induced by the summation map π . Since $H_b^0(G, W_0(G, X)^{**}) = (W_0(G, X)^{**})^G$ and $H_b^0(G, \mathbb{R}) = \mathbb{R}$ we have that $[J] = 0$ if and only if there exists an element $\mu \in W_0(G, X)^{**}$ such that $\mu = g\mu$ and $\mu(\pi) = 1$. Thus μ is an invariant mean for the action.

We turn to the implication (2) implies (4). Let μ be the invariant mean associated with the action. For every $h \in G$ and for every equivariant bounded cochain ϕ we define $s_h\phi : G^p \rightarrow \mathcal{E}^*$ by $s_h\phi(g_0, \dots, g_{p-1}) = \phi(g, g_0, \dots, g_{p-1})$; we note that for fixed h , $s_h\phi$ is not equivariant in general. However, the map s_h does satisfy the identity $ds_h + s_hd = 1$ for every $h \in G$, and we will now construct an equivariant contracting homotopy, adapting an averaging procedure introduced in [4].

For $\phi \in C_b^p(G, \mathcal{E}^*)$ let $\widehat{\phi} : G^p \rightarrow \ell^\infty(G, \mathcal{E}^*)$ be defined by $\widehat{\phi}(\mathbf{g})(h) = s_h\phi(\mathbf{g})$, for $\mathbf{g} = (g_0, \dots, g_{p-1})$.

Note that for every $k, h \in G$,

$$\begin{aligned} \widehat{\phi}(kg_0, \dots, kg_{p-1})(h) &= \phi(h, kg_0, \dots, kg_{p-1}) = k(\phi(k^{-1}h, g_0, \dots, g_{p-1})) \\ &= k(\widehat{\phi}(g_0, \dots, g_{p-1})(k^{-1}h)) \\ &= (k(\widehat{\phi}(g_0, \dots, g_{p-1}))(h)) \end{aligned}$$

so $\widehat{\phi}(k\mathbf{g}) = k(\widehat{\phi}(\mathbf{g}))$.

We can now define a map $s : C^p(G, \mathcal{E}^*) \rightarrow C^{p-1}(G, \mathcal{E}^*)$:

$$s\phi(\mathbf{g}) = \mu_{\mathcal{E}}(\widehat{\phi}(\mathbf{g})),$$

where $\mu_{\mathcal{E}} : \ell^\infty(G, \mathcal{E}^*) \rightarrow \mathcal{E}^*$ is the map defined in Lemma 19 using the invariant mean μ . Note that $\|\mu_{\mathcal{E}}\| \leq 2\|\mu\|$, and $\|\widehat{\phi}(\mathbf{g})\| \leq \sup\{\|\phi(\mathbf{k})\| \mid \mathbf{k} \in G^{p+1}\}$. Hence $s\phi$ is bounded.

For every cochain ϕ , $k(s\phi) = s(k\phi) = s\phi$ since $\widehat{\phi}$ and $\mu_{\mathcal{E}}$ are equivariant.

The map s provides a contracting homotopy for the complex $C_b^*(G, \mathcal{E}^*)$ which can be seen as follows. As $\mu_{\mathcal{E}} : \ell^\infty(G, \mathcal{E}^*) \rightarrow \mathcal{E}^*$ is a linear operator it follows that for a given $\phi \in C_b^p(G, \mathcal{E}^*)$, and a $p + 1$ -tuple of arguments $\mathbf{k} = (k_0, \dots, k_p)$, $ds\phi$ is obtained by applying the mean $\mu_{\mathcal{E}}$ to the map $g \mapsto ds_g\phi(\mathbf{k})$, while $sd\phi$ is obtained by applying $\mu_{\mathcal{E}}$ to the function $g \mapsto s_gd\phi(\mathbf{k})$. Thus

$$(sd + ds)\phi(\mathbf{k}) = \mu_{\mathcal{E}}(g \mapsto (ds_g + s_gd)\phi(\mathbf{k})).$$

Given that $ds_g + s_gd = 1$ for every $g \in G$, for every $\mathbf{g} \in G^{p+1}$ the function $g \mapsto (ds_g + s_gd)\phi(\mathbf{k}) = \phi(\mathbf{k}) \in \mathcal{E}^*$ is constant, and so by Lemma 19,

$$(sd + ds)\phi(\mathbf{k}) = (ds_e + s_ed)\phi(\mathbf{k}) = \phi(\mathbf{k}).$$

Thus $sd + ds = 1$, as required.

Collecting these results together, we have proved that (2) implies (4).

The fact that (4) implies (3), follows from the fact that $N_0(G, X)^*$ is an ℓ^1 -geometric G - $C(X)$ -module, proved in Lemma 8. \square

REFERENCES

- [1] S. Adams, Boundary amenability for word hyperbolic groups and an application to smooth dynamics of simple groups. *Topology* 33 (1994), no. 4, 765–783.
- [2] C. Anantharaman-Delaroche, J. Renault, *Amenable groupoids*, Monographies de L'Enseignement Mathématique, 36. L'Enseignement Mathématique, Geneva, 2000.
- [3] J. Block and S. Weinberger, Aperiodic tilings, positive scalar curvature and amenability of spaces. *J. Amer. Math. Soc.* 5 (1992), no. 4, 907–918.
- [4] J. Brodzki, G. A. Niblo and N. J. Wright, A cohomological characterisation of Yu's Property A for metric spaces. Preprint, University of Southampton ePrints November 2009: arXiv:1002.5040.
- [5] N. P. Brown, N. Ozawa, *C*-algebras, and finite dimensional approximations*, Graduate Studies in Mathematics, 88, AMS, Rhode Island 2008.
- [6] R. G. Douglas and P. W. Nowak, Invariant expectations and vanishing of bounded cohomology for exact groups. *Journal of Topology and Analysis* 3 (2011), no. 1, 89–107.
- [7] E. Guentner, J. Kaminker, Exactness and the Novikov conjecture. *Topology* 41 (2002), no. 2, 411–418.
- [8] U. Hamenstädt, Geometry of the mapping class groups. I. Boundary amenability. *Invent. Math.* 175 (2009), no. 3, 545–609.
- [9] B. E. Johnson, *Cohomology of Banach Algebras*, Memoirs of the AMS Number 127, 1972, AMS, Providence, Rhode Island.
- [10] N. Higson, J. Roe, Amenable group actions and the Novikov conjecture, *J. Reine Angew. Math.* 519 (2000), 143–153.
- [11] R. E. Megginson, *An Introduction to Banach Space Theory*, (Graduate Texts in Mathematics 183), Springer Verlag New York, 1998.
- [12] N. Monod, A note on topological amenability. arXiv:1004.0199v2
- [13] N. Ozawa, Amenable actions and exactness for discrete groups, *C.R. Acad. Sci. Paris Ser. I Math* 330 (2000), 691–695.
- [14] G. Yu, The Coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space, *Inventiones Math.* 139 (2000), 201–240.

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