

# COARSELY EMBEDDABLE METRIC SPACES WITHOUT PROPERTY A

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ABSTRACT. We study Guoliang Yu's Property A and construct metric spaces which do not satisfy Property A but embed coarsely into the Hilbert space.

## 1. INTRODUCTION

Guoliang Yu introduced a weak version of amenability for discrete metric spaces, which he called Property A [Yu]. This property, if satisfied for a metric space  $X$ , implies the existence of a coarse embedding of  $X$  into the Hilbert space. For metric spaces with bounded geometry this implies that the Coarse Baum-Connes Conjecture and in a particular case when this space is a finitely generated group  $\Gamma$  with the word length metric, the Novikov Conjecture for  $\Gamma$  [Yu].

The converse, whether every coarsely embeddable metric space has Property A was not known. The only examples of spaces which are known so far not to satisfy Property A are expanders and Gromov's groups which contain them in their Cayley graphs [Gr<sub>2</sub>],  $\ell_p$ -spaces for  $2 < p \leq \infty$  [JR], [DGLY], box spaces (see [Roe<sub>1</sub>]) and warped cones [Roe<sub>2</sub>]. Only in the last two cases methods other than non-embeddability into  $\ell_2$  were developed to show that Property A is not satisfied, however in all the known cases these spaces also do not admit a coarse embedding into the Hilbert space.

In this paper we study Property A and its behavior for locally finite metric spaces. The main idea, roughly speaking, is to look at the smallest, diameter of the support with which Property A is satisfied for a given, usually bounded, metric space. Our main observation is that for the  $n$ -fold direct products of amenable groups this best diameter must grow to infinity with  $n$ . This allows us to construct metric spaces which do not have Property A. More precisely, our main example is a disjoint union  $\coprod \Gamma^n$ , where  $\Gamma$  is a finite group. The reasoning we use is flexible enough not to obstruct coarse embeddability and thus our examples embed coarsely into the Hilbert space.

For background we refer the reader to [Roe<sub>1</sub>] for a self-contained, thorough treatment of coarse geometry, in particular discussion of Property A and coarse embeddability, and to [Pier] and [BHV, Appendix G] for a survey of amenability.

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## 2. PROPERTY A

In what follows whenever  $A$  is a set,  $|A|$  will denote its cardinality. A discrete metric space  $X$  is locally finite if  $|B(x, R)| < \infty$  for every  $x \in X$  and  $R \in \mathbb{R}$ .  $X$  has bounded geometry if for every  $R > 0$  there exists a number  $\mathcal{N}(R) > 0$  such that for every  $x \in X$  we have

$$|B(x, R)| \leq \mathcal{N}(R).$$

A locally finite, in particular a bounded geometry metric space is necessarily countable.

**Definition 2.1** ([Yu]). *A discrete metric space  $X$  has Property A if for every  $R > 0$  and  $\varepsilon > 0$  there is a collection  $\{\mathcal{A}_x\}_{x \in X}$  of finite subsets of  $X \times \mathbb{N}$  and  $S > 0$  such that*

- (1)  $(x, 1) \in \mathcal{A}_x$  for every  $x \in X$
- (2)  $\frac{|\mathcal{A}_x \Delta \mathcal{A}_y|}{|\mathcal{A}_x \cap \mathcal{A}_y|} \leq \varepsilon$  when  $d(x, y) \leq R$ ;
- (3)  $\mathcal{A}_x \subset B(x, S) \times \mathbb{N}$

The class of finitely generated groups possessing Property A is quite large - so far the only such groups known not to have Property A are Gromov's groups which contain expanders in their Cayley graphs. There are also groups for which it is not yet known whether they have Property A, e.g. Thompson's group  $F$ .

It was also shown by Guentner, Kaminker and Ozawa that a finitely generated group has Property A if and only if the reduced group  $C^*$ -algebra  $C_r^*(\Gamma)$  is exact, see [Oz].

We will need the weak Reiter's condition, a reformulation of Property A in terms of finitely supported functions in the unit sphere of the Banach space  $\ell_1$ . This was proved by Higson and Roe [HR].

Denote

$$\ell_1(X)_{1,+} = \{f \in \ell_1(X) \mid \|f\|_1 = 1, f \geq 0\}.$$

In other words,  $\ell_1(X)_{1,+}$  is the space of positive probability measures on  $X$ . If  $\Gamma$  is a finitely generated group,  $\gamma \in \Gamma$  and  $f \in \ell_1(\Gamma)_{1,+}$  then by  $\gamma \cdot f$  we denote the translation of  $f$  by element  $\gamma$ , i.e.

$$(\gamma \cdot f)(g) = f(\gamma^{-1}g).$$

**Proposition 2.2** ([HR]). *Let  $X$  be a discrete metric space with bounded geometry. The following conditions are equivalent:*

- (1)  $X$  has property A;
- (2) For every  $R > 0$  and  $\varepsilon > 0$  there exists a map  $\xi : X \rightarrow \ell_1(X)_{1,+}$  and  $S \in \mathbb{R}$  such that  $\|\xi(x) - \xi(y)\|_1 \leq \varepsilon$  whenever  $d(x, y) \leq R$  and  $\text{supp } \xi(x) \subseteq B(x, S)$  for every  $x \in X$ .

It will become important in the last section that the assumption of bounded geometry is needed only when proving (2) $\Rightarrow$ (1).

### 3. PROPERTY A AND AMENABLE GROUPS

A finitely generated group  $\Gamma$  will be always considered with a left-invariant integer-valued metric  $d_\Gamma$  (e.g. word length metric) which takes all values between 0 and the diameter of the group. The length of an element  $\gamma \in \Gamma$  is defined to be  $|\gamma| = d_\Gamma(\gamma, e)$ . We also use the standard notation  $B_\Gamma(S)$  to denote a ball of radius  $S$  around the identity, we will most times omit the subscript  $\Gamma$  if it does not lead to confusion.

**Definition 3.1.** Let  $X$  be a discrete metric space.

(A) For a given  $R > 0$ ,  $\varepsilon > 0$  and map  $\xi : X \rightarrow \ell_1(X)_{1,+}$  satisfying

$$(1) \quad \|\xi(x) - \xi(y)\|_1 \leq \varepsilon$$

for every  $x, y \in X$  such that  $d(x, y) \leq R$ , denote

$$S_X(\xi, R, \varepsilon) = \inf S,$$

$S_X(\xi, R, \varepsilon) \in \mathbb{N} \cup \{\infty\}$ , where the infimum is taken over all  $S > 0$  satisfying  $\text{supp } \xi(x) \subseteq B(x, S)$  for every  $x \in X$ .

(B) Define

$$\text{diam}_X^{\mathcal{A}}(R, \varepsilon) = \inf S_X(\xi, R, \varepsilon),$$

$\text{diam}_X^{\mathcal{A}}(R, \varepsilon) \in \mathbb{N} \cup \{\infty\}$ , where the infimum is taken over all maps  $\xi : X \rightarrow \ell_1(X)_{1,+}$  satisfying (1) with the given  $R$  and  $\varepsilon$  for all  $x, y \in X$  such that  $d(x, y) \leq R$ .

(C) If  $\Gamma$  is a finitely generated group then for given  $R > 0$ ,  $\varepsilon > 0$  by  $\text{diam}_\Gamma^{\mathcal{F}}(R, \varepsilon) \in \mathbb{N} \cup \{\infty\}$  we denote the smallest  $S$  such that there exists a function  $f \in \ell_1(\Gamma)_{1,+}$  such that  $\text{supp } f \subseteq B(S)$  and

$$(2) \quad \|f - \gamma \cdot f\|_1 \leq \varepsilon$$

for all  $\gamma \in \Gamma$  such that  $|\gamma| \leq R$ .

Thus  $\text{diam}_\Gamma^{\mathcal{F}}$  is the notion resulting from restricting (A) and (B) to considering only functions  $\xi : \Gamma \rightarrow \ell_1(\Gamma)_{1,+}$  satisfying (1) from Definition 3.1 (A), and which are translates of a single function  $f \in \ell_1(\Gamma)_{1,+}$ , i.e.  $\xi(\gamma) = \gamma \cdot f$  for every  $\gamma \in \Gamma$  and for some fixed  $f \in \ell_1(X)_{1,+}$ .

The exact values of both  $\text{diam}_X^{\mathcal{A}}$  and  $\text{diam}_\Gamma^{\mathcal{F}}$  depend on the metric, in particular in the case of a word length on the group, on the choice of the generating set. What is independent of such choices is whether  $\text{diam}_X^{\mathcal{A}}$  and  $\text{diam}_\Gamma^{\mathcal{F}}$  are finite or infinite. The

following is a straight-forward consequence of Definition 3.1 and the Proposition 2.2.

**Proposition 3.2.** (1a) *If a discrete metric space  $X$  has Property A then  $\text{diam}_X^{\mathcal{A}}(R, \varepsilon) < \infty$  for every  $R > 0$  and  $\varepsilon > 0$ .*

(1b) *A discrete metric space  $X$  with bounded geometry has Property A if and only if  $\text{diam}_X^{\mathcal{A}}(R, \varepsilon) < \infty$  for every  $R > 0$  and  $\varepsilon > 0$ .*

(2) *A finitely generated group is amenable if and only if  $\text{diam}_\Gamma^{\mathcal{F}}(R, \varepsilon) < \infty$  for every  $R > 0$  and  $\varepsilon > 0$ .*

We will make use of the fact that Definition 3.1 gives nontrivial notions for bounded metric spaces. Such a space, call it  $X$ , has Property A for any  $R$  and  $\varepsilon$  with  $S = \text{diam } X$ , however in general  $\text{diam}_X^{\mathcal{A}}(R, \varepsilon)$  might be drastically smaller than  $\text{diam } X$ . We also have  $\text{diam}_X^{\mathcal{A}}(R, \varepsilon) \leq \text{diam}_X^{\mathcal{A}}(R', \varepsilon')$  whenever  $R' \leq R$  and  $\varepsilon \leq \varepsilon'$ , and similar inequalities hold for  $\text{diam}_\Gamma^{\mathcal{F}}$ .

As mentioned before, a significant class of discrete spaces with Property A is given by finitely generated amenable groups. What we are interested in is how in this case  $\text{diam}_\Gamma^{\mathcal{A}}$  behaves and whether it is related to  $\text{diam}_\Gamma^{\mathcal{F}}$ .

Recall that one of the definitions of amenability provides the existence of a left-invariant mean on  $\ell_\infty(\Gamma)$  (see e.g. [BHV, Appendix G] for a survey of amenability), that is of a positive, normalized, left-invariant linear functional on  $\ell_\infty(\Gamma)$ . For a finite group  $\Gamma$  and  $\xi : \Gamma \rightarrow \mathbb{R}$  the mean of  $f$  is given by

$$\int_\Gamma f(\gamma) d\sigma(\gamma),$$

where  $d\sigma$  is the normalized Haar measure on  $\Gamma$ . For an amenable group  $\Gamma$  and  $\xi \in \ell_\infty(\Gamma)$  we will denote the mean of  $f$  by

$$\int_\Gamma f(g) dg.$$

**Theorem 3.3.** *Let  $\Gamma$  be finitely generated amenable group and fix  $R \geq 1$ ,  $\varepsilon > 0$ . Then*

$$\text{diam}_\Gamma^{\mathcal{A}}(R, \varepsilon) = \text{diam}_\Gamma^{\mathcal{F}}(R, \varepsilon).$$

*Proof.* To show the inequality  $\text{diam}_\Gamma^{\mathcal{A}}(R, \varepsilon) \leq \text{diam}_\Gamma^{\mathcal{F}}(R, \varepsilon)$ , given a finitely supported function  $f \in \ell_1(\Gamma)_{1,+}$  satisfying 2 from Definition 3.1 (C) for  $R > 0$  and  $\varepsilon > 0$  and all  $\gamma \in \Gamma$  such that  $|g| \leq R$ , consider the map  $\xi : \Gamma \rightarrow \ell_1(\Gamma)_{1,+}$  defined by  $\xi(\gamma) = \gamma \cdot f$ .

To prove the other inequality assume that  $\Gamma$  satisfies conditions from (2) of Proposition 2.2 for  $R > 0$ ,  $\varepsilon > 0$  with  $S > 0$  realized by the function  $\xi : \Gamma \rightarrow \ell_1(\Gamma)_{1,+}$ . For every  $\gamma \in \Gamma$  define

$$f(\gamma) = \int_\Gamma \xi(g)(\gamma^{-1}g) dg.$$

This gives a well-defined function  $f : \Gamma \rightarrow \mathbb{R}$ ,  $\xi(g)(\gamma^{-1}g)$  as a function of  $g$  belongs to  $\ell_\infty(\Gamma)$  since  $\xi(g)(\gamma) \leq 1$  for all  $\gamma, g \in \Gamma$ .

First observe that if  $|\gamma| > S$  then  $\xi(g)(\gamma^{-1}g) = 0$  for all  $g \in \Gamma$ , thus  $f(\gamma) = 0$  whenever  $|\gamma| > S$ . Consequently,

$$\begin{aligned} \|f\|_1 &= \sum_{\gamma \in B(S)} f(\gamma) = \sum_{\gamma \in B(S)} \int_{\Gamma} \xi(g)(\gamma^{-1}g) dg \\ &= \int_{\Gamma} \left( \sum_{\gamma \in B(S)} \xi(g)(\gamma^{-1}g) \right) dg = \int_{\Gamma} 1 dg = 1. \end{aligned}$$

Thus  $f$  is an element of  $\ell_1(\Gamma)_{1,+}$ .

If  $\lambda \in \Gamma$  is such that  $|\lambda| \leq R$  then

$$\begin{aligned} \|f - \lambda \cdot f\|_{\ell_1(\Gamma)} &= \sum_{\gamma \in \Gamma} |f(\gamma) - f(\lambda^{-1}\gamma)| \\ &= \sum_{\gamma \in B(S) \cup \lambda B(S)} \left| \int_{\Gamma} \xi(g)(\gamma^{-1}g) dg - \int_{\Gamma} \xi(g)((\lambda^{-1}\gamma)^{-1}g) dg \right| \\ &= \sum_{\gamma \in B(S) \cup \lambda B(S)} \left| \int_{\Gamma} \xi(g)(\gamma^{-1}g) dg - \int_{\Gamma} \xi(\lambda^{-1}g)(\gamma^{-1}g) dg \right| \\ &= \sum_{\gamma \in B(S) \cup \lambda B(S)} \left| \int_{\Gamma} (\xi(g)(\gamma^{-1}g) - \xi(\lambda^{-1}g)(\gamma^{-1}g)) dg \right| \\ &\leq \int_{\Gamma} \left( \sum_{\gamma \in B(S) \cup \lambda B(S)} |\xi(g)(\gamma^{-1}g) - \xi(\lambda^{-1}g)(\gamma^{-1}g)| \right) dg \\ &\leq \int_{\Gamma} \varepsilon dg = \varepsilon, \end{aligned}$$

since

$$\begin{aligned} \int_{\Gamma} \xi(g)((\lambda^{-1}\gamma)^{-1}g) dg &= \int_{\Gamma} \lambda \cdot (\xi(g)((\lambda^{-1}\gamma)^{-1}g)) dg \\ &= \int_{\Gamma} \xi(\lambda^{-1}g)(\gamma^{-1}g) dg, \end{aligned}$$

by the left invariance of the mean.

Thus for the previously chosen  $R$  and  $\varepsilon$  we have constructed a function  $f \in \ell_1(\Gamma)_{1,+}$  satisfying  $\|f - \gamma \cdot f\|_{\ell_1(\Gamma)} \leq \varepsilon$  whenever  $1 \leq |\gamma| \leq R$  and  $\text{supp } f \subseteq B(S)$  for the same  $S$  as for  $\xi$ . This proves the second inequality.  $\square$

#### 4. BEHAVIOR OF PROPERTY A IN HIGH-DIMENSIONAL

## PRODUCTS OF AMENABLE GROUPS

Let  $(X_1, d_{X_1}), (X_2, d_{X_2})$  be metric spaces. We will consider the cartesian product  $X_1 \times X_2$  with the  $\ell_1$ -metric, i.e.

$$d_{X_1 \times X_2}(x, y) = d_{X_1}(x_1, y_1) + d_{X_2}(x_2, y_2),$$

for  $x = (x_1, x_2), y = (y_1, y_2)$ , both in  $X_1 \times X_2$ . If  $\Gamma_1, \Gamma_2$  are finitely generated groups such metric on  $\Gamma_1 \times \Gamma_2$  is left-invariant if and only if the metrics on the factors are. In particular, if the metric on the factors is the word length metric then the  $\ell_1$ -metric on the direct product gives the word length metric associated to the standard set generators arising from the generators on the factors.

In this section we study how does  $\text{diam}^{\mathcal{A}}$  behave for cartesian powers of a fixed finitely generated amenable group  $\Gamma$ . Theorem 3.3 will be our main tool, allowing us to reduce questions about  $\text{diam}_{\Gamma^n}^{\mathcal{A}}$  to questions about  $\text{diam}_{\Gamma^n}^{\mathcal{F}}$ . Note that if  $X$  and  $Y$  are discrete metric spaces, and for every  $R > 0$  and  $\varepsilon > 0$  there are maps  $\xi : X \rightarrow \ell_1(X)$  and  $\zeta : Y \rightarrow \ell_1(Y)$  realizing Property A for  $X$  and  $Y$  respectively, then the maps  $\xi \otimes \zeta : X \times Y \rightarrow \ell_1(X \times Y)$  of the form

$$\xi \otimes \zeta(x, y) = \xi(x) \zeta(y),$$

give Property A for  $X \times Y$  in the sense of Proposition 2.2 and in the particular case when  $Y = X$  the diameter of the supports increases (the reader can extract precise estimates from [DG]). The main result of this section shows that this is always the case.

**Proposition 4.1.** *Let  $\Gamma$  be a finitely generated group and assume*

$$\|f - \gamma \cdot f\|_1 \geq \varepsilon_0$$

for all  $f \in \ell_1(\Gamma)_{1,+}$  with  $\text{supp } f \subseteq B_\Gamma(S) \subseteq \Gamma$  where  $S > 0$  is fixed and  $|\gamma| = 1$ . Then for any  $n \in \mathbb{N}$  and  $f \in \ell_1(\Gamma^n)_{1,+}$  with  $\text{supp } f \subseteq B_{\Gamma^n}(S) \subseteq \Gamma^n$  and any  $\gamma \in \Gamma^n$ ,  $|\gamma| = 1$ ,

$$\|f - \gamma \cdot f\|_1 \geq \varepsilon_0.$$

*Proof.* Let  $f \in \ell_1(\Gamma \times \Gamma)_{1,+}$  and  $f_x : \{x\} \times \Gamma \rightarrow \mathbb{R}$  be the restriction of  $f$  to the set  $\{x\} \times \Gamma$ . Then for  $\gamma \in \Gamma \times \{e\}$ ,

$$\begin{aligned} \|f - \gamma \cdot f\|_1 &= \sum_{x \in \Gamma} \|f_x - \gamma \cdot f_x\|_1 = \sum_{x \in \Gamma} \frac{\|f_x - \gamma \cdot f_x\|_1}{\|f_x\|_1} \|f_x\|_1 \\ &\geq \varepsilon_0 \sum_{x \in \Gamma} \|f_x\|_1 = \varepsilon_0. \end{aligned}$$

Similarly we prove the claim for  $\gamma \in \{e\} \times \Gamma$ . The claim for  $\Gamma^n$  follows by induction on  $n$ .  $\square$

**Proposition 4.2.** *Let  $\Gamma$  be a finitely generated group and  $f_n$  be a sequence of functions  $f_n \in \ell_1(\Gamma)_{1,+}$ , such that*

$$\|f_n - \gamma \cdot f_n\|_1 < 2,$$

and

$$\text{supp } f_n \subseteq B_{\Gamma^n}(S)$$

for a fixed  $S \in \mathbb{N}$ , some  $R > 0$  and all  $\gamma \in \Gamma^n$  with  $|\gamma| \leq R$ . Then  $|\text{supp } f_n| \rightarrow \infty$ .

*Proof.* Assume the contrary, that  $|\text{supp } f_n| \leq N \in \mathbb{N}$  for infinitely many  $n \in \mathbb{N}$ . Choose any  $n \geq SN + 1$ . Then  $\text{supp } f_n \subseteq \Gamma^N \subsetneq \Gamma^n$  so for any  $\gamma \notin \Gamma^N$ ,  $|\gamma| = 1$  we have

$$\|f - \gamma \cdot f\|_1 = 2.$$

□

**Remark 4.3.** It follows from the above proof that the number of elements in the support of  $f_n$  is bounded below by  $\frac{n}{S}$ . Intuitively one can expect that the supports of the functions  $f_n$  will be "thick" and "evenly distributed" in all the dimensions, so in general we believe one should have much better estimates. Compare however [HR, Lemma 4.3].

The next theorem is the key ingredient in the construction of spaces without Property A.

**Theorem 4.4.** *Let  $\Gamma$  be a finitely generated amenable group. Then for any  $0 < \varepsilon < 2$ ,*

$$\liminf_{n \rightarrow \infty} \text{diam}_{\Gamma^n}^{\mathcal{F}}(1, \varepsilon) = \infty.$$

*Proof.* Assume the contrary. Then there exists an  $S \in \mathbb{N}$  such that for infinitely many  $n \in \mathbb{N}$  there is a function  $f_n \in \ell_1(\Gamma^n)_{1,+}$  satisfying

$$\|f_n - \gamma \cdot f_n\|_1 \leq \varepsilon,$$

$\text{supp } f_n \subset B_{\Gamma^n}(S)$  for all  $\gamma \in \Gamma$  such that  $|\gamma| = 1$ . Fix  $\delta \leq \frac{2-\varepsilon}{2S}$  and  $m \in \mathbb{N}$  and for any  $n \in \mathbb{N}$  for which  $f_n$  as above exists consider the decomposition

$$\Gamma^n = \Gamma^m \times \Gamma^m \times \dots \times \Gamma^m \times \Gamma^r$$

where  $0 \leq r < m$ . For  $k = 1, \dots, \frac{n-r}{m}$  denote by  $\partial_k f_n$  the restriction of  $f_n$  to the set

$$\{g \in \text{supp } f_n : |g| = S, g_{(i)} \neq e \Leftrightarrow (k-1)m + 1 \leq i \leq km\},$$

of those elements of  $\text{supp } f_n$  whose length in this  $k$ -th factor  $\Gamma^m$  is exactly  $S$ , and extend it with 0 to a function on the whole  $\Gamma^n$ ; we denote by  $g_{(i)}$  the  $i$ -th coordinate of  $g \in \Gamma^n$  as an element of the cartesian product.

Since for  $k \neq l$ , where  $km + r \leq n$  and  $lm + r \leq n$ , we have

$$\text{supp } \partial_k f_n \cap \text{supp } \partial_l f_n = \emptyset$$

and

$$\sum_{k=1}^{\frac{n-r}{m}} \|\partial_k f_n\|_1 \leq \|f_n\|_1 = 1,$$

we can conclude that for every  $\hat{\varepsilon} > 0$ , which we now choose to satisfy  $\frac{\varepsilon+2\hat{\varepsilon}}{1-\hat{\varepsilon}} \leq \varepsilon+\delta$ , there exists a sufficiently large  $n \in \mathbb{N}$  and  $i \in \mathbb{N}$  such that

$$\|\partial_i f_n\|_1 \leq \hat{\varepsilon}.$$

Denote

$$\varphi = \frac{f_n - \partial_i f_n}{\|f_n - \partial_i f_n\|_1} \in \ell_1(X)_{1,+}.$$

We have

$$\begin{aligned} \|\varphi - \gamma \cdot \varphi\|_1 &= \frac{\|(f_n - \gamma \cdot f_n) + (\gamma \cdot \partial_i f_n - \partial_i f_n)\|_1}{\|f_n - \partial_i f_n\|_1} \\ &\leq \frac{\varepsilon + 2\hat{\varepsilon}}{1 - \hat{\varepsilon}} \leq \varepsilon + \delta, \end{aligned}$$

by the previous choice of  $\hat{\varepsilon}$ .

Now consider the decomposition  $\Gamma^n = \Gamma^m \times \Gamma^{n-m}$  where  $\Gamma^m$  is the  $i$ -th factor in which we performed the previous operations on  $f_n$ . For every  $g \in \Gamma^m$  define (we're recycling the letter  $f$  here, the "old"  $f$ 's don't appear in the proof anymore)

$$f(g) = \sum_{h \in \Gamma^{n-m}} \varphi(gh),$$

where  $h \in \Gamma^{n-m}$ . Then  $f \in \ell_1(\Gamma^m)_{1,+}$  and  $\text{supp } f \subseteq B_{\Gamma^m}(S - 1)$ . Moreover, for an element  $\gamma \in \Gamma^m$  of length 1,

$$\begin{aligned} \|f - \gamma \cdot f\|_1 &= \sum_{g \in \Gamma^m} |f(g) - f(\gamma^{-1}g)| \\ &= \sum_{g \in \Gamma^m} \left| \sum_{h \in \Gamma^{n-m}} \varphi(gh) - \varphi(\gamma^{-1}gh) \right| \\ &\leq \sum_{g \in \Gamma^m} \sum_{h \in \Gamma^{n-m}} |\varphi(gh) - \varphi(\gamma^{-1}gh)| \\ &= \sum_{g \in \Gamma^n} |\varphi(g) - \varphi(\gamma^{-1}g)| = \|\varphi - \gamma \cdot \varphi\|_1 \leq \varepsilon + \delta. \end{aligned}$$

Since  $m \in \mathbb{N}$  was arbitrary we can obtain a family  $\{f_m\}_{m \in \mathbb{N}}$  of functions  $f_m \in \ell_1(\Gamma^m)_{1,+}$  satisfying

$$\|f_m - \gamma \cdot f_m\|_1 \leq \varepsilon + \delta$$

and  $\text{supp } f_m \subseteq B_{\Gamma^m}(S - 1)$  where  $\delta$  is independent of  $m$ . If we apply the procedure described above to this family we can again reduce the diameter of the supports of the functions  $f_m$  and obtain yet another new family  $\{f_m\}_{m \in \mathbb{N}}$  of functions  $f_m \in \ell_1(\Gamma^m)_{1,+}$  such that

$$\|f_m - \gamma \cdot f_m\|_1 \leq \varepsilon + 2\delta$$

and  $\text{supp } f_m \subseteq B_{\Gamma^m}(S - 2)$ .



After repeating this procedure  $S$  times we obtain a family  $\{f_m\}_{m \in \mathbb{N}}$  such that  $f \in \ell_1(\Gamma^m)_{1,+}$  and

$$\begin{aligned} \|f_m - \gamma \cdot f_m\|_1 &\leq \varepsilon + S\delta \\ &\leq \varepsilon + S \frac{2-\varepsilon}{2S} < 2, \end{aligned}$$

since  $\delta \leq \frac{2-\varepsilon}{2S}$ . However, for every  $m \in \mathbb{N}$

$$f_m(g) = \begin{cases} 1 & \text{when } g = e, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\|f_m - \gamma \cdot f_m\| = 2$$

for every  $m \in \mathbb{N}$  and every  $\gamma \in \Gamma^m$ , which gives a contradiction.  $\square$

**Remark 4.5.** In the proofs in this section we have reduced the study Property A to studying amenability, however we expect that the above considerations can be carried out as well in a more general setting for the price of complicating the arguments and estimates.

## 5. CONSTRUCTING EMBEDDABLE SPACES WITHOUT PROPERTY A

In this section we construct metric spaces which do not have Property A. The idea is natural: take a disjoint union of bounded, locally finite metric spaces, for which it is known that they satisfy Property A with diameters growing to infinity, so that we violate the condition from Proposition 3.2.

On the other hand the condition  $\text{diam}_X^{\mathcal{A}}(R, \varepsilon) = \infty$  for any  $R > 0$  and  $\varepsilon > 0$  does not rule out coarse embeddability into the Hilbert space, which is characterized by the existence of a  $c_0$ -type functions in the sphere of  $\ell_1$ . This was proved by Dadarlat and Guentner [DG], see also [No<sub>1</sub>] for discussion and applications.

Given a sequence  $\{(X_n, d_n)\}_{n=1}^{\infty}$  we will make the disjoint sum  $X = \coprod X_n$  into a metric space by giving it a metric  $d_X$  such that

- (1)  $d_X$  restricted to  $X_n$  is  $d_n$ ,
- (2)  $d_X(X_n, X_{n+1}) \geq n + 1$ ,
- (3) if  $n \leq m$  we have  $d_X(X_n, X_m) = \sum_{k=n}^{m-1} d_X(X_k, X_{k+1})$ .

**Theorem 5.1.** *Let  $\Gamma$  be a finite, group. The (locally finite) metric space  $X_{\Gamma} = \coprod_{n=1}^{\infty} \Gamma^n$  has the following properties:*

- (1)  $X_{\Gamma}$  does not have Property A
- (2)  $X_{\Gamma}$  embeds coarsely into  $\ell_p$  for any  $1 \leq p \leq \infty$ .

*Proof.* To prove 1) observe that by 3.2 if  $\mathcal{X}_\Gamma$  would satisfy Property A then  $\text{diam}_{\mathcal{X}_\Gamma}^{\mathcal{A}}(1, \varepsilon)$  would be finite for every  $0 < \varepsilon < 2$ , which in turn would imply that the restriction of maps  $\xi$  realizing Property A for every  $\varepsilon$  and  $R = 1$  to each  $\Gamma^n \subseteq \mathcal{X}_\Gamma$  gives Property A with diameter bounded uniformly in  $n$ ,

$$\sup_{n \in \mathbb{N}} \text{diam}_{\Gamma^n}^{\mathcal{A}}(1, \varepsilon) < \infty,$$

since  $B_{\mathcal{X}_\Gamma}(x, R) = B_{\Gamma^n}(x, R)$  for all sufficiently large  $n$  and all  $x \in \Gamma^n \subset \mathcal{X}_\Gamma$ . However by theorems 4.4 and 3.3,

$$\text{diam}_{\Gamma^n}^{\mathcal{F}}(1, \varepsilon) = \text{diam}_{\Gamma^n}^{\mathcal{A}}(1, \varepsilon)$$

and

$$\text{diam}_{\Gamma^n}^{\mathcal{F}}(1, \varepsilon) \longrightarrow \infty$$

as  $n \rightarrow \infty$ .

To prove 2), note that since  $\Gamma$  is a finite metric space any one-to-one map from  $\Gamma$  into the space  $\ell_1$  is biLipschitz. Denote the biLipschitz constant by  $L$ . Then the product map

$$f^n = \underbrace{f \times f \times \dots \times f}_{n \text{ times}} : \Gamma^n \rightarrow \left( \sum_{i=1}^n \ell_1 \right)_1$$

is also a biLipschitz map with the same constant  $L$ , where  $\left( \sum_{i=1}^n \ell_1 \right)_1$  denotes a direct sum of  $n$  copies of  $\ell_1$  with a  $\ell_1$ -metric, which is of course isometrically isomorphic to  $\ell_1$ . It is clear that this suffices to embed  $\mathcal{X}_\Gamma$  into  $\ell_1$  coarsely.

In [No<sub>1</sub>] the author proved that the Hilbert space embeds coarsely into any  $\ell_p$ ,  $1 \leq p \leq \infty$  and that the properties of coarse embeddability into  $\ell_p$  for  $1 \leq p \leq 2$  are all equivalent. Thus  $\mathcal{X}_\Gamma$  embeds coarsely into the Banach space  $\ell_p$  for any  $1 \leq p \leq \infty$ .  $\square$

Note that in the simplest case  $G = \mathbb{Z}_2$ , the space  $\mathcal{X}_{\mathbb{Z}_2}$  is a disjoint union of discrete cubes of increasing dimensions, with the  $\ell_1$ -metric. We thus have the following

**Corollary 5.2.** *An infinite-dimensional cube complex does not have Property A*

On the other hand it is also not hard to construct an infinite-dimensional cube complex which embeds coarsely into any  $\ell_p$ , giving a different realization of examples discussed above.

We also want to mention a conjecture formulated by Dranishnikov [Dr, Conjecture 4.4] that a discrete metric space  $X$  has Property A if and only if  $X$  embeds coarsely into the space  $\ell_1$ . The examples discussed in this section are in particular counterexamples to Dranishnikov's conjecture.

**Remark 5.3.** As it is apparent from the proof, the theorem works as soon as  $\Gamma$  is amenable and admits a quasi-isometric embedding into  $\ell_1$ . This is satisfied e.g. for finitely generated abelian groups, however it is not clear what happens in the case of other amenable groups and whether they all embed quasi-isometrically into  $\ell_1$ .

Definition 3.1 suggests to study asymptotics of growth functions related to Property A in the spirit of Følner functions introduced by Vershik, or equivalently, isoperimetric profiles as defined by Gromov. Such invariants are studied in [No<sub>2</sub>].

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