ISOPERIMETRY OF GROUP ACTIONS

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ABSTRACT. The isoperimetric profile of a discrete group was introduced by Vershik, however it is well defined only for a restrictive class amenable groups. We generalize the notion of isoperimetric profile beyond the world of amenable groups by defining isoperimetric profiles of amenable actions of finitely generated groups on compact topological spaces. This allows to extend the definition of the isoperimetric profile to all groups which are exact in such a way that for amenable groups it is equal to Vershik's isoperimetric profile. The main feature of our construction is that is preserves many of the properties known from the classical case. We use these results to compute exact asymptotics of the isoperimetric profiles for several classes of non-amenable groups.

1. INTRODUCTION AND DISCUSSION OF THE RESULTS

The study of isoperimetric problems has a long history in geometry and has been used extensively in different settings. Invariants based on isoperimetry such as isoperimetric dimension or isoperimetric profiles have become one of the most important and basic tools, as they have many connections and applications. Such invariants are well defined provided that the manifold is *regularly exhaustible*. This condition guarantees the existence of a sequence of open sets $U \subset \mathcal{M}^n$ with smooth boundaries such that

$$\frac{\mathrm{Vol}_{n-1}(\partial U)}{\mathrm{Vol}_n(U)} \longrightarrow 0.$$

One defines the isoperimetric profile by minimizing the number $\mathrm{Vol}_n(U)$ over all U for which the above ratio is smaller than $\frac{1}{k}$ for $k = 1, 2, \ldots$. The asymptotics of this function is a quasi-isometry invariant and allows to obtain information about the large-scale geometry of the manifold.

In this article we are interested in isoperimetry for finitely generated groups which was introduced in a similar way by Vershik in [45]. The isoperimetric profile of a finitely group is defined exactly as above (see section 2.3 for a precise definition), where volume is understood as the number of elements in the set and the boundary of a set F consists of those elements in the complement of F which are at distance 1 from F. Again however we have to assume that there exists a sequence of finite sets in the group for which $\frac{\#\partial F}{\#F} \to 0$ and this greatly restricts the class of groups for which the isoperimetric profile is well defined. This class is exactly the class of amenable groups defined by von Neumann as the ones possessing

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an invariant mean and it is a rather small class among all finitely generated groups, since already groups with free subgroups do not belong to it.

The two approaches come together on infinite, regular covers of compact manifolds. In that case the cover is exhaustible if and only if the group of deck transformations is amenable and the two objects are quasi-isometric. This in particular means that the finding the isoperimetric profile on the cover is, up to constant factors, equivalent to finding the one of the group of deck transformations. A case of special interest is the one of the universal cover which is quasi-isometric to the fundamental group of the manifold.

The main purpose of this article is to extend the notion of the isoperimetric profile in a meaningful way beyond the world of amenable groups, namely to all groups which have Property A, or in other words are C^*-exact. This last class is incomparably larger than the class of amenable groups and contains almost all known groups. In particular our isoperimetric profile is well defined for a lot of groups with Kazhdan's Property (T), which were inaccessible for the previous definition. In fact at present it is only known that groups which do not satisfy Property A exist and finding explicit examples of such groups is a wide open question of geometric group theory.

In order to generalize the notion of isoperimetric profile we first introduce the isoperimetric profile of an amenable group action. The notion of amenable action was introduced by Zimmer in ergodic theory [49] and since then has appeared repeatedly in various contexts. We use here the notion of a topologically amenable action which was defined in [2]. With our definition of an isoperimetric profile of an amenable action Vershik's original isoperimetric profile is simply the isoperimetric profile of the action on a point. We show that the profile of an amenable action shares with its predecessor a number of natural properties such as estimates for subgroups and direct products as well as invariance under conjugacy and independence of the asymptotics on the generating set.

With this definition we move on to define the isoperimetric profile of a group with Property A to be the isoperimetric profile of the canonical action of G on βG, the Stone-Čech compactification of G. This definition does not require any auxiliary space X with an action of G and it also has the property that it minimizes the isoperimetric profile over all amenable actions of a given group G. This greater generality causes however difficulties and in particular it is not clear whether the isoperimetric profile of an exact group is a quasi-isometry invariant for all non-amenable groups. We can nevertheless deduce invariance for finite index subgroups and under quasi-isometries for a class of groups whose isoperimetric profiles are *lossless* with respect to the isodiametric profile. This last class includes all the examples in which the isoperimetric profiles are computed.

In order to estimate the isoperimetric profile we generalize some previously known tools used in the context of Vershik's profiles of amenable groups. The first step is to relate the isoperimetric profile of an action to Vershik's profile. We prove that for an action with an invariant mean the profile of the action is equal to the profile of the trivial action. This in particular shows that our definition is indeed the correct generalization.

A well-known theorem of Coulhon and Saloff-Coste [15] states that the volume growth of a group gives a lower bound on the isoperimetric profile. Our second step is a generalization of this theorem. The main tool here is a Sobolev-type inequality for finitely supported functions on groups with values in a G-C^*-algebra. From this inequality we can derive a satisfying generalization of the theorem of Coulhon and Saloff-Coste: the isoperimetric profile of any action is bounded from below by volume growth of the group.

The third and last method used to estimate the isoperimetric profile is inspired by [34] and gives an upper bound on the profile in terms of type of asymptotic dimension. Asymptotic dimension is a coarse counterpart of covering dimension and its type measures the diameters of elements of covers appearing in the definition of asymptotic dimension. The same estimate holds for the isoperimetric profiles of groups with Property A, generalizing the main theorem of [34].

The above techniques allow to determine completely the generalized isoperimetric profiles of some finitely generated groups, most of which are non-amenable, showing that they are exactly exponential. Among these we have hyperbolic groups, Baumslag-Solitar groups, Coxeter groups, as well as groups acting properly, cocompactly by isometries on certain widely used spaces.

This paper is a natural continuation of [34] and some proofs here follow the ideas of that paper. In particular the averaging theorem for Property A was used by the author in various context in [34, 35, 37]. Also, another approach to isoperimetry of actions is presented in [23]. To finish the introduction we would like to pose three questions which seem to be important from the point of view of the present article.

Question 1.1. *Is the generalized isoperimetric profile a quasi-isometry invariant for all groups with Property A?*

Question 1.2. *Is the isoperimetric profile lossless with respect to the isodiametric profile* A_G *(see Remark 4.18) for every group with Property A?*

Question 1.3. *Does Erschler's formula* $\mathrm{Føl}_{G\wr H} \simeq (\mathrm{Føl}_G)^{\mathrm{Føl}_H}$ *hold for all groups with Property A?*

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2. Preliminaries

2.1. **Asymptotics.** Given a function $f : \mathbb{N} \to \mathbb{N}$ it will be convenient to view it as a piecewise linear function $f : \mathbb{R} \to \mathbb{R}$, determined by it's values on the integers. This is so because we are only interested in the asymptotics of such functions and we don't loose any generality in this way.

Given two such functions $f, g : \mathbb{N} \to \mathbb{N}$ we write $f \preceq g$ if $f(n) \leq C_1 g(C_2 n)$ for some constants C_1, C_2, and $f \simeq g$ if $f \preceq g$ and $g \preceq f$. We will write $f \prec g$ if the inequality is strict, i.e. $f \preceq g$ but it is not true that $f \simeq g$. We will also often say that f is linear if $f(n) \preceq n$, polynomial if $f(n) \preceq n^k$ for some $k \in \mathbb{N}$ and so on.

We will sometimes write the inverse f^{-1} of a function for which it is not clear if it has an actual inverse. What we mean by this is the inverse of an invertible function that has the same asymptotics as f.

2.2. **Groups and Volume Growth.** Given a finitely generated group G we will always denote a finite generating set by S and indicate the choice by writing $(G; S)$. Moreover we will always assume that S is symmetric, that is $s \in S \implies s^{-1} \in S$. The identity element is denoted by e.

We always view $(G; S)$ as a metric space with the word length metric associated to S. The length of an element $g \in G$ will be then denoted $|g|_S$. By $B_{(G;S)}(x, r)$ we will denote the ball of radius $r \geq 0$ centered at a point $x \in G$, i.e.

$$B_{(G;S)}(x, r) = \{y \in G : |x^{-1}y| \leq r\}.$$

For a set A by $\#A$ we denote the cardinality of A. The volume growth function $\mathrm{Vol}_{(G;S)} : \mathbb{N} \to \mathbb{N}$ is defined by setting

$$\mathrm{Vol}_{(G;S)}(n) = \#B_{(G;S)}(e, n).$$

We will also define

$$\Theta_{(G;S)}(k) = \min\{n : \mathrm{Vol}_{(G;S)}(n) \geq k\},$$

the inverse of the volume growth function. In all of the above we will drop the subscripts if it is clear which group or generating set we are referring to.

2.3. **Amenability and Isoperimetry.** Given a group acting on a compact space by homeomorphisms, an invariant mean for the action is a positive, normalized functional on $C(X)$ which is also invariant under the action of G. We will denote the value of the mean by $\int_X f\, dx$ for $f \in C(X)$. A group is amenable if the action on $\ell_\infty(G)$ has an invariant mean. Følner characterized amenability in terms of an isoperimetric condition: a group is amenable if for every $\varepsilon > 0$ there exists a finite set $F \subset G$ such that

$$\frac{\#\partial F}{\#F} \leq \varepsilon,$$

where $\partial F = \{g \in G \setminus F : d(g, F) = 1\}$. Typical references on amenability are [4, Appendix G], [21]. Vershik proposed in [45] to use Følner's characterization to define the isoperimetric profile of an amenable group, following the case of the Riemannian manifold.

Definition 2.1 (**Vershik's isoperimetric profile**, [45])**.** *Let $(G; S)$ be an amenable group. Isoperimetric profile (also called the Følner function) is the function* $\mathrm{Føl}^*_{(G;S)} : \mathbb{N} \to \mathbb{N}$ *defined by setting*

$$\mathrm{Føl}^*_{(G;S)}(n) = \min \# F$$

where the minimum is taken over all finite sets $F \subset G$ such that $\frac{\#\partial F}{\# F} \leq \frac{1}{n}$.

We will slightly change the notation later on, see the remarks following Proposition 3.3. The notion of an isoperimetric profile can also be defined in a different way so that it would make sense for all groups, however in that case it gives non-trivial information only for amenable groups. To do that set for instance

$$I_G(n) = \max_{m \leq n} \min_{F \subset G, \# F = n} \# \partial F,$$

and observe that non-amenability is equivalent to linearity of I_G. One of the main features of isoperimetric profiles is the following

Proposition 2.2. *Let G and H be quasi-isometric finitely generated groups. Then*

$$\mathrm{Føl}^*_G \simeq \mathrm{Føl}^*_H .$$

Because of this one usually omits the reference to the generating set. Isoperimetric profiles have been studied extensively and have appeared in several contexts. In geometric analysis isoperimetric profiles can be interpreted in terms of Sobolev inequalities. They were also linked to random walks by Varopoulos ([44, 16]), who used the isoperimetric profile to estimate the decay of the heat kernel, or in other words, the probability of the return to the origin of the simple random walk on the Cayley graph of an amenable group. We refer the reader to [39, 41] and the references therein.

In [34] isoperimetric profiles have also been linked to asymptotic dimension and allowed to find examples of finitely generated groups with finite asymptotic dimension but infinite Assouad-Nagata dimension.

The function $\mathrm{Føl}^*$ appears also in Riemannian geometry. Take a compact Riemannian manifold $\mathcal{N}$ and a covering space $\mathcal{M}$ associated to a normal subgroup $G \leq \pi_1(\mathcal{N})$. The group of deck transformations of $\mathcal{M}$ is the quotient $\pi_1(\mathcal{N})/G$ and solving the isoperimetric problem on $\mathcal{M}$ is equivalent to finding the asymptotics of $\mathrm{Føl}^*_{\pi_1(\mathcal{N})/G}$. We refer to [42] for details. Direct computations of isoperimetric profiles can be found in e.g. [19, 24, 31, 39].

3. Isoperimetric profiles of actions on compact spaces

3.1. **Amenable actions and isoperimetry.**

We recall the definition of a topologically amenable action below, to do that we need some notation. Given a compact, Hausdorff space X we consider the C^*-algebra $C(X)$ of continuous, complex-valued functions on X with the supremum norm and identity element $\mathbb{1}_X$. Denote $C(X)_+ = \{f \in C(X) : f(x) \geq 0 \text{ for all } x \in X\}$, the set of all non-negative elements of $C(X)$. For a function $\xi : G \to C(X)$ we define the support of ξ by

$\operatorname{supp}\xi = \{g \in G : \xi_g \neq 0\}$. For a given group G and compact space X most of our calculations will take place in the Banach module

$$\ell_1(G; C(X)) = \left\{\xi : G \to C(X) : \Big\| \sum_{g\in G} |\xi_g| \Big\|_{C(X)} < \infty \right\}$$

with the norm

$$\|\xi\|_{\ell_1(G;C(X))} = \Big\| \sum_{g\in G} |\xi_g| \Big\|_{C(X)},$$

where $|f|$ denotes the absolute value of $f \in C(X)$. We write ξ_g to denote the image of g under ξ in $C(X)$, i.e. the $C(X)$-coefficient of g.

Assume now that a group G acts by homeomorphisms on a compact Hausdorff space X. This action induces an action of G by automorphisms on $C(X)$, denoted $(\gamma * f)(x) = f(\gamma^{-1}x)$ for every $\gamma \in G$, $f \in C(X)$. Combining the above action with the action of G on itself by translations we define an action of G on $\ell_1(G; C(X))$:

$$(\gamma \cdot \xi)_g = \gamma * \xi_{\gamma^{-1}g},$$

where $\xi : G \to C(X)$ and $g, \gamma \in G$. Now we recall the definition of an amenable action due to Anantharaman-Delaroche and Renault [2].

Definition 3.1. *Let G be a finitely generated group acting on a compact topological space by homeomorphisms. The action is amenable if for every $\varepsilon > 0$ there exists a finitely supported function $\xi : G \to C(X)_+$ such that*

(1) $\sum_{g\in G} \xi_g = \mathbb{1}_X$,
(2) $\|\xi - s \cdot \xi\|_{\ell_1(G;C(X))} \leq \varepsilon$ *for every* $s \in S$.

Thus a group is amenable if and only if its action on a point is amenable. Amenable actions, as a natural generalization of amenability appear in various contexts. The most recent and important appearance is in connection with C^*-exactness and Property A of discrete groups. Property A is a geometric, Følner-type condition introduced by Guoliang Yu, see [48, 36]. It guarantees coarse embeddability of a metric space (such as a finitely generated group) into the Hilbert space, and through Yu's work, the Coarse Baum-Connes Conjecture. The latter, through non-vanishing of higher indices of various differential operators, implies the Novikov conjecture, the zero-in-the-spectrum conjecture and non-existence of metric of uniformly positive scalar curvature [48]. It was shown by Higson and Roe that a finitely generated group has Property A if and only if it admits an amenable action on some compact space [28].

On the other hand Kirchberg was considering the notion of exactness of C^*-algebras, defining a C^*-algebra exact if the minimal tensor product with A preserves short exact sequences. The work of Guentner and Kaminker [27] and Ozawa [38] showed that exactness of the reduced group C^*-algebra is equivalent to Property A. We will define the notion of Property A later, in section 4.

The following definition introduces the notion of isoperimetry of an amenable action on a compact space.

Definition 3.2 (**Isoperimetric profile of an amenable action**). *Let X be a compact topological space and let $(G; S)$ be a group with a finite symmetric generating set S acting on X amenably. We define the isoperimetric profiles (or Følner function) of the given action* $\mathrm{Føl}_{(G;S)\curvearrowright X} : \mathbb{N} \to \mathbb{N}$ *by setting*

$$\mathrm{Føl}_{(G;S)\curvearrowright X}(n) = \inf \ \#\operatorname{supp}\xi,$$

where the infimum is taken over all finitely supported $\xi : G \to C(X)_+$ satisfying conditions of Definition 3.1 with $\varepsilon = \frac{1}{n}$.

We will focus mainly on two cases: when X is a point and when X is the Stone-Čech compactification of the group G with its natural action. The first case is addressed below, the second in Section 4. First we observe several properties which hold for a general X.

Proposition 3.3. *Let G be a finitely generated group. Then*

$$\mathrm{Føl}_{(G;S)\curvearrowright\{\mathrm{pt}\}} \simeq \mathrm{Føl}^*_{(G;S)}.$$

The proof is standard, we leave the details to the reader and refer to [4, 21] for necessary background. From now on we will abuse the notation introduced in Definition 2.1 and denote $\mathrm{Føl}^*_{(G;S)}(n) = \mathrm{Føl}_{(G;S)\curvearrowright\{\mathrm{pt}\}}(n)$. As before the dependance on the generating set must be encoded in the definition, however the asymptotics of Føl of a fixed action action do not depend on the generating set.

Proposition 3.4. *Let G be a group and S and T be two finite generating sets of G. Then*

$$\mathrm{Føl}_{(G;S)\curvearrowright X} \simeq \mathrm{Føl}_{(G;T)\curvearrowright X}.$$

Proof. We have $L^{-1}|g|_T - C \leq |g|_S \leq L|g|_T + C$ and since the generator are exactly the elements of length 1 in the word length metric, the result follows from applying the triangle inequality multiple times. ∎

From now on we will omit the reference to the generating set whenever it is unnecessary. We now show that the isoperimetric profile has strong invariance properties with respect to continuous equivariant maps, and in particular, conjugation.

Proposition 3.5. *Let $(G; S)$ be a group acting on compact spaces X and Y and let $F : X \to Y$ be a continuous, equivariant map (i.e., $F(gx) = g(F(x))$ for all $g \in G$). Then*

$$\mathrm{Føl}_{(G;S)\curvearrowright X}(n) \leq \mathrm{Føl}_{(G;S)\curvearrowright Y}(n)$$

for all $n \in \mathbb{N}$.

Proof. Let $\eta : G \to C(Y)_+$ be a function satisfying condition of Definition 3.1 for $\varepsilon > 0$. Define $\xi : G \to C(X)_+$ by the equality

$$\xi_g(x) = \eta_g(f(x))$$

for all $g \in G$ and $x \in X$. We clearly have $\sum_{g\in G} \xi_g = \mathbb{1}_X$ and for $s \in S$

$$\begin{aligned}
\|\xi - s\cdot\xi\|_{\ell_1(G;C(X))} &= \Big\|\sum_{g\in G} |\eta_g(F(x)) - \eta_{s^{-1}g}(F(s^{-1}x))|\Big\|_{C(X)} \\
&= \Big\|\sum_{g\in G} |\eta_g(F(x)) - \eta_{s^{-1}g}(s^{-1}F(x))|\Big\|_{C(X)} \\
&\le \varepsilon.
\end{aligned}$$

Finally, $\operatorname{supp}\xi \subseteq \operatorname{supp}\eta$, which proves the claim. $\square$

Let G act on spaces X and Y. The two actions are conjugate if the continuous map $f : X \to Y$ in the formulation of the above theorem is a homeomorphism.

Corollary 3.6. *If G acts on compact spaces X and Y and the actions are conjugate then* $\mathrm{Føl}_{(G;S)\curvearrowright X}(n) = \mathrm{Føl}_{(G;S)\curvearrowright Y}(n)$ *for every $n \in \mathbb{N}$.*

Below we list two applications of the above proposition. Let G be a group acting on X. Then G acts on X^k for any $k \in \mathbb{N}$ via a diagonal action.

Corollary 3.7. *For any G acting on X we have* $\mathrm{Føl}_{(G;S)\curvearrowright X}(n) = \mathrm{Føl}_{(G;S)\curvearrowright X^k}(n)$ *for any $k, n \in \mathbb{N}$.*

Proof. The diagonal map $\Delta : X \to X^k$ and the projection $\pi : X^k \to X$ to any of the factors are both continuous and equivariant maps and the claim follows from Proposition 3.5. $\square$

Corollary 3.8. (1) *Let $\{X_i, \varphi_i\}_{i\in\mathbb{N}}$ be a direct system of compact G-spaces with equivariant bonding maps φ_i. Then*

$$\sup_i \ \mathrm{Føl}_{G\curvearrowright X_i}(n) \le \mathrm{Føl}_{G\curvearrowright \varinjlim X_i}(n)$$

(2) *Let $\{X_i, \varphi_i\}_{i\in\mathbb{N}}$ be an inverse system of compact G-spaces with equivariant bonding maps φ_i. Then*

$$\mathrm{Føl}_{G\curvearrowright \varprojlim X_i}(n) \le \inf_i \ \mathrm{Føl}_{G\curvearrowright X_i}(n)$$

One of the fundamental properties of the isoperimetric profile is the estimate for subgroups [19, Lemma 4]. For isoperimetric profiles of actions the same estimate holds.

Proposition 3.9. *Let G, H be finitely generated groups acting on a compact space X and let H be a subgroup of G. Let Ω be a generating set of H and S be a generating set of G such that $\Omega \subseteq S$. Then*

$$\mathrm{Føl}_{(H;\Omega)\curvearrowright X}(n) \le \mathrm{Føl}_{(G;S)\curvearrowright X}(n)$$

for every $n \in \mathbb{N}$. In particular $\mathrm{Føl}_{H\curvearrowright X} \preceq \mathrm{Føl}_{G\curvearrowright X}$.

Proof. Consider $\xi : G \to C(X)_+$ satisfying the conditions of Definition 3.1 for a given $\varepsilon > 0$. Consider the right cosets of H in G given by $H\gamma$ where for each coset we have a one chosen representative γ. Denote by C the set of such γ's. Define

$$\eta_h = \sum_{\gamma\in C} \xi_{h\gamma}.$$

Observe that η is finitely supported, $\eta_h \geq 0$ for all $h \in H$ and

$$\sum_{h\in H} \eta_h = \sum_{h\in H}\sum_{\gamma\in C} \xi_{h\gamma} = \sum_{g\in G} \xi_g = \mathbb{1}_X.$$

Also if we let $s \in \Omega$ then

$$\begin{aligned}
\|\eta - s\cdot\eta\|_{\ell_1(H;C(X))} &= \Big\| \sum_{h\in H} |\eta_h - s * \eta_{s^{-1}h}| \Big\|_{C(X)} \\
&= \Big\| \sum_{h\in H} \Big| \sum_{\gamma\in C} \xi_{h\gamma} - s * \xi_{s^{-1}h\gamma} \Big| \Big\|_{C(X)} \\
&\leq \Big\| \sum_{g\in G} |\xi_g - \xi_{s^{-1}g}| \Big\|_{C(X)} \\
&= \|\xi - s\cdot\xi\|_{\ell_1(G;C(X)} \leq \varepsilon.
\end{aligned}$$

Finally observe that by construction $\#\operatorname{supp}\eta \leq \#\operatorname{supp}\xi$ and the claim follows. □

Certain estimates for asymptotics of the isoperimetric profiles of products of groups are known, see e.g. [14]. Here we have a generalization of such estimates to group actions.

Proposition 3.10. *Let G, H be finitely generated groups acting amenably on compact spaces X and Y respectively. Then*

$$\mathrm{Føl}_{G\times H\curvearrowright X\times Y} \preceq (\mathrm{Føl}_{G\curvearrowright X})(\mathrm{Føl}_{H\curvearrowright Y}).$$

Proof. Let $\xi : G \to C(X)$ and $\eta : H \to C(Y)$ satisfy Definition 3.1 for a given $\varepsilon > 0$. Define the function $\alpha : G\times H \to C(X\times Y)$ by

$$\alpha_{(g,h)} = \xi_g \dot{\otimes} \eta_h,$$

where

$$\left(\xi_g \dot{\otimes} \eta_h\right)(x,y) = \xi_g(x)\eta_h(y).$$

Note that if either $\xi_g = 0$ or $\eta_h = 0$ then $\xi_g \dot{\otimes} \eta_h = 0$, so $\#\operatorname{supp}\alpha \leq (\#\operatorname{supp}\xi)(\#\operatorname{supp}\eta)$. The group $G\times H$ is generated by the set $S_G\times\{e\}\cup\{e\}\times S_H$. So take an element of this generating set, say $\sigma = (s,e)$. Then

$$\begin{aligned}
\|\alpha - \sigma\cdot\alpha\|_{\ell_1(G\times H;C(X\times Y))} &= \Big\| \sum_{(g,h)} |\xi_g \dot{\otimes} \eta_h - (s * \xi_{s^{-1}g}) \dot{\otimes} \eta_h| \Big\|_{C(X\times Y)} \\
&= \Big\| \sum_h \sum_g \Big| \eta_h \dot{\otimes} \left(\xi_g - s * \xi_{s^{-1}g}\right) \Big| \Big\|_{C(X\times Y)} \\
&\leq \Big\| \sum_h |\eta_h| \, \|\xi - s\cdot\xi\|_{\ell_1(G;C(X))} \Big\|_{C(X\times Y)} \\
&\leq \varepsilon.
\end{aligned}$$

The proof for a generator of the form (e,s) is similar and the claim follows. □

3.2. **Relation to actions on a point.** We will investigate how does the Følner function of an action relate to the Følner function of a trivial action. The next statement is a special case of Proposition 3.5 but we will point it out as a separate lemma. It states that the action on a point has the worst isoperimetry out of all actions of a given group.

Lemma 3.11. *Let G be a finitely generated group acting on a compact space X. Then* $\mathrm{Føl}_{(G;S)\curvearrowright X}(n) \leq \mathrm{Føl}^*_{(G;S)}(n)$ *for all* $n \in \mathbb{N}$.

Proof. For any action of G on X the map $X \to \{\mathrm{pt}\}$ is equivariant and we apply Proposition 3.5. □

In certain cases the converse of the above proposition holds as well. First let G be a group acting on a compact space X with a fixed point. Then observe that $\mathrm{Føl}^*_{(G;S)}(n) = \mathrm{Føl}_{(G;S)\curvearrowright X}(n)$ for all $n \in \mathbb{N}$. In particular such an action is amenable if and only if G is amenable. To see this observe that the inclusion of the fixed point of the action is an equivariant map and the claim follows from Proposition 3.5. Thus, for instance, amenable actions of Property (T) groups never have fixed points. Another way to view the above statement is to note that the functional on $C(X)$ obtained by taking the value at the fixed point is an invariant mean for the action and generalization of this idea this is the subject of the next theorem. The proof exploits the averaging procedure for amenable actions that was used earlier by the author in [34, 35].

Theorem 3.12 (**Reduction to action on a point**). *Assume that the action of G on X has an invariant mean. Then*

$$\mathrm{Føl}_{(G;S)\curvearrowright X}(n) = \mathrm{Føl}^*_{(G;S)}(n) \tag{1}$$

for every $n \in \mathbb{N}$. In particular, if G is amenable then (1) holds for any action on a compact space.

Proof. In view of lemma 3.11 we need to prove $\mathrm{Føl}_{(G;S)\curvearrowright X}(n) \geq \mathrm{Føl}^*_{(G;S)}(n)$. Consider a function $\xi : G \to C(X)$ satisfying both conditions in Definition 3.1 with a given ε and define a function $\eta : G \to [0, \infty)$ by the formula

$$\eta_g = \int_X \xi_g \, dx$$

for every $g \in G$. Observe that $\operatorname{supp} \eta \subseteq \operatorname{supp} \xi$ and $\eta_g \geq 0$ for every $g \in G$. We compute the norm of η:

$$\begin{aligned} \sum_{g \in G} \eta_g &= \sum_{g \in G} \int_X \xi_g \, dx \\ &= \int_X \left(\sum_{g \in G} \xi_g \right) dx \\ &= \int_X \mathbb{1}_X \, dx = 1. \end{aligned}$$

We also have the following estimate on the norm of $\eta - s \cdot \eta$:

$$
\begin{aligned}
\|\eta - s \cdot \eta\|_{\ell_1(G)} &= \sum_{g \in G} |\eta_g - \eta_{s^{-1}g}| \\
&= \sum_{g \in G} \left| \int_X \xi_g \, dx - \int_X \xi_{s^{-1}g} \, dx \right| \\
&= \sum_{g \in G} \left| \int_X \xi_g \, dx - \int_X s * \xi_{s^{-1}g} \, dx \right| \\
&\le \sum_{g \in G} \int_X |\xi_g - s * \xi_{s^{-1}g}| \, dx \\
&= \int_X \|\xi - s \cdot \xi\|_{\ell_1(G;C(X))} \, dx \;\le\; \varepsilon,
\end{aligned}
$$

by the invariance of the mean (3rd line) and finiteness of the sum above. This shows that $\mathrm{Føl}_{(G;S)\curvearrowright X}(n) \ge \mathrm{Føl}^*_{(G;S)}(n)$ and the asserted equality follows.

The second part of the theorem follows from the fact that any action of an amenable group has an invariant mean. □

Note also that Theorem 3.12 gives a criterion for non-amenability. To show that a group G is non-amenable it is enough to find two actions of G on compact spaces which have different isoperimetric profiles.

3.3. **Sobolev-type inequalities and volume growth.** The first lower bound on the isoperimetric profile was given by Kaimanovich in terms of the spectral measure [30]. Later Coulhon and Saloff-Coste [15] showed that there exists a general estimate from below on the isoperimetric profile, namely they proved that the function $\mathrm{Føl}^*_G$ grows faster than Vol_G. This fact turned out to be extremely useful and having defined isoperimetric profiles of actions one of the first questions is whether the same type of estimate holds for these profiles. It turns out that this is indeed the case and the estimate holds in a much broader sense than the classical case, allowing to change the coefficients from $\mathbb{C}$ to any C^*-algebra with a group action (see also Remark 3.14).

The main result will be derived from the following Sobolev-type inequality, the reader is referred to [39] for the formulation and proof of the original inequality.

Proposition 3.13 (**Coulhon-Saloff-Coste inequality with coefficients**). *Let $(G;S)$ be a finitely generated group acting on a compact topological space X by homeomorphisms. Then the inequality*

$$\|f\|_{\ell_1(G;C(X))} \;\le\; 2\Theta_{(G;S)}\,(2\,\#\operatorname{supp} f)\, \sup_{s \in S} \|f - s \cdot f\|_{\ell_1(G;C(X))},$$

holds for every finitely supported function $f : G \to C(X)$.

Proof. Let $f : G \to C(X)$ be a finitely supported function and denote

$$\delta_f = \sup_{s \in S} \|f - s \cdot f\|_{\ell_1(G;C(X))}.$$

Observe that for any element $\gamma \in G$ we have

$$\|f - \gamma \cdot f\|_{\ell_1(G;C(X))} \leq |\gamma|\, \delta_f, \tag{2}$$

by applying the triangle inequality multiple times. Let κ be the smallest n such that

$$\mathrm{Vol}_{(G;S)}(n) \geq 2\#\operatorname{supp} f. \tag{3}$$

So, in other words, $\kappa = \Theta_{(G;S)}(2\#\operatorname{supp} f)$. From (2) we have

$$\begin{aligned}
\kappa\, \delta_f &\geq \frac{1}{\mathrm{Vol}_{(G;S)}(\kappa)} \sum_{|\gamma| \leq \kappa} \|f - \gamma \cdot f\|_{\ell_1(G;C(X))} \\
&= \frac{1}{\mathrm{Vol}_{(G;S)}(\kappa)} \sum_{|\gamma| \leq \kappa} \Big\| \sum_{x \in G} \left| f_x - \gamma * f_{\gamma^{-1}x} \right| \Big\|_{C(X)} \\
&\geq \frac{1}{\mathrm{Vol}_{(G;S)}(\kappa)} \Big\| \sum_{|\gamma| \leq \kappa} \sum_{x \in G} \left| f_x - \gamma * f_{\gamma^{-1}x} \right| \Big\|_{C(X)}. \qquad (4)
\end{aligned}$$

Since the double sum above is finite we can change the order of summation and we chose κ as in (3), we see that for any $x \in G$ at least half of the points in the ball $B(x, \kappa)$ is not in the support of f. Thus for any $x \in G$ and at least half of γ's satisfying $|\gamma| \leq \kappa$ we have

$$f_{\gamma^{-1}x} = 0, \quad \text{and consequently} \quad \left| f_x - \gamma * f_{\gamma^{-1}x} \right| = |f_x|.$$

This gives us the following inequality between elements of $C(X)$:

$$\sum_{x \in G} \sum_{|\gamma| \leq \kappa} \left| f_x - \gamma \cdot f_{\gamma^{-1}x} \right| \geq \frac{\mathrm{Vol}_{(G;S)}(\kappa)}{2} \sum_{x \in G} |f_x|.$$

Plugging this back into (4) we obtain

$$\begin{aligned}
\kappa\, \delta_f &\geq \frac{1}{\mathrm{Vol}_{(G;S)}(\kappa)} \left\| \frac{\mathrm{Vol}_{(G;S)}(\kappa)}{2} \sum_{x \in G} |f_x| \right\|_{C(X)} \\
&= \frac{1}{2} \|f\|_{\ell_1(G;C(X))}.
\end{aligned}$$

The claim is proved. □

The inequality in [39] is slightly different and it does not fit into the context of this article. In particular in [39], as in most Sobolev-type inequalities, a version of the gradient is used. Observe that in the above inequality the number δ_f is closely related to the gradient and serves the same purpose.

Remark 3.14. Note that nowhere in the proof of Proposition 3.13 have we used commutativity of the C^*-algebra $C(X)$ and it is only a formality to replace in Proposition 3.13 the topological space X with the action of G by homeomorphisms by *any* C^*-algebra (neither necessarily commutative nor unital) with an action of G by $*$-automorphisms. In general one can also consider a noncommutative, unital C^*-algebra instead of $C(X)$ in Definition 3.1, however this way we will not fulfill the requirements of an amenable action on a noncommutative C^*-algebra, it is still an open problem how to define such amenable actions so that appropriate facts about crossed products would follow from the definition, see [1] for discussion.

We now prove the main result of this section.

Theorem 3.15 (**Lower bound by volume growth**). *Let G be a finitely generated group acting amenably on a topological space X. Then*

$$\text{Føl}_{G\curvearrowright X} \geq \text{Vol}_G\,.$$

Proof. Assume that we're given a finitely supported function $\xi : G \to C(X)$ satisfying conditions of Definition 3.1 for a given $\varepsilon = \frac{1}{n}$. Then $\sup_{s\in S} \|\xi - s\cdot\xi\|_{\ell_1(G;C(X))} \leq \frac{1}{n}$ and since $\|\xi\|_{\ell_1(G;C(X))} = 1$, Proposition 3.13 yields

$$\Theta_{(G;S)}(2\,\#\operatorname{supp}\xi) \geq \frac{n}{2}.$$

Since $k \geq \text{Vol}_{(G;S)}(\Theta_{(G;S)}(k) - 1)$ and Vol_G is an increasing function we obtain

$$\begin{aligned} \#\operatorname{supp}\xi &\geq \frac{1}{2}\,\text{Vol}_{(G;S)}\left(\Theta_{(G;S)}(2\,\#\operatorname{supp}\xi) - 1\right) \\ &\geq \frac{1}{2}\,\text{Vol}_{(G;S)}\left(\frac{n}{2} - 1\right) \end{aligned}$$

and the claim follows. □

4. Generalized isoperimetric profiles of groups

4.1. **Isoperimetry of groups and universal spaces.** In this section we extend Vershik's Følner function in such a way that it makes sense for all exact groups. By Theorem 3.12 the isoperimetric profile of every amenable action on a compact space generalizes Vershik's profile, however we want to obtain independence of any prescribed auxiliary space X. In order to do this for a given group G we will take the isoperimetric profile of a canonical action of G. An example of such is the action on a point, but as we have seen it is the most restrictive one and reserved only for amenable groups. We will go to the other extreme and take the largest possible space on which G acts canonically, namely the Stone-Čech compactification βG.

Definition 4.1. *Let X be a topological space. The Stone-Čech compactification βX is a compact Hausdorff space with the following two properties:*

(1) *X is a dense open subset of βX,*
(2) *for any continuous map $f : X \to K$ where K is compact and Hausdorff, there exists a unique extension $\overline{f} : \beta X \to K$.*

One can also characterize βG algebraically, $C(\beta X)$ is naturally isomorphic to $C_b(X)$, the space of bounded continuous functions on X. The group G acts on βG by homeomorphisms in the following way. Denote the left translation by an element $g \in G$ by L_g. There exists an extension $\overline{L}_g : \beta G \to \beta G$ such that the diagram

$$\begin{tikzcd} G \arrow[r, "L_g"] \arrow[d, "i"] & G \arrow[d, "i"] \\ \beta G \arrow[r, "\overline{L}_g"'] & \beta G \end{tikzcd}$$

is commutative. We will not introduce the usual definitions of Property A and refer the reader to [46] for an extensive survey. The following theorem of Higson and Roe will serve as the definition of Property A.

Theorem 4.2 ([28])**.** *Let G be a finitely generated group. The following conditions are equivalent:*

(1) *G has Property A;*
(2) *G acts amenably on βG.*

The following definition is the second main definition in this article. It generalizes the isoperimetric profile of an amenable group to all groups which have Yu's Property A.

Definition 4.3. *Let G be a group which satisfies Property A. Define the isoperimetric profile* $\text{Føl}_{(G;S)} : \mathbb{N} \to \mathbb{N}$ *by the formula*

$$\text{Føl}_{(G;S)}(n) = \text{Føl}_{(G;S)\curvearrowright\beta G}(n)$$

for every $n \in \mathbb{N}$.

By Proposition 3.4 we can again drop the reference to the generating set when discussing asymptotics. The class of groups with Property A includes amenable groups, hyperbolic groups [48], linear groups [26], groups acting on finite dimensional cube complexes [13],Coxeter groups [17], extensions and free products of the above and many more. In fact the only groups that are known so far not to have Property A are Gromov's random groups containing expanders in their Cayley graphs, so definition 4.3 indeed broadens significantly the domain of isoperimetric profile. In particular Kazhdan's Property (T), which is a very strong version of non-amenability, does not exclude Property A. Thus Definition 4.3 makes groups with Property (T) accessible to the isoperimetric profile.

Let us now thoroughly explain the choice $X = \beta G$ in Definition 4.3. Recall that lemma 3.11 stated that the action on a point has the worst isoperimetric profile among all possible actions. Here on the other hand, by the universal property of βG the action on βG has the best isoperimetric profile among all amenable actions.

Proposition 4.4. *Let G be a finitely generated group. Then*

$$\text{Føl}_{(G;S)}(n) = \min \text{Føl}_{(G;S)\curvearrowright X}(n),$$

where the minimum is taken over all amenable actions of G on compact Hausdorff spaces X.

Proof. To prove this fact assume that G admits an amenable action on a compact space X. The map $\theta : G \to X$ sending G to one of the orbits of G is continuous, thus by the universal property of the Stone-Čech compactification there exists a continuous, equivariant extension of this map $f : \beta G \to X$. By Proposition 3.5, $\text{Føl}_{(G;S)\curvearrowright \beta G}(n) \leq \text{Føl}_{(G;S)\curvearrowright X}(n)$ and the claim follows. □

On the other hand, if G is amenable it is irrelevant what X is, since by Theorem 3.12 $\text{Føl}_{(G;S)\curvearrowright \beta G}$ reduces to Vershik's function $\text{Føl}^*_{(G;S)}$. Thus it is clear that Definition 4.3 extends Vershik's definition. The following is yet another generalization of [19, Lemma 4].

Proposition 4.5. *Let G, H be finitely generated groups and let H be a subgroup of G. Let Ω be a generating set of H and S be a generating set of G such that $\Omega \subseteq S$. Then*

$$\text{Føl}_{(H;\Omega)}(n) \leq \text{Føl}_{(G;S)}(n)$$

for every $n \in \mathbb{N}$. In particular $\text{Føl}_H \leq \text{Føl}_G$.

Proof. By Proposition 3.9 we have $\text{Føl}_{(H;\Omega)\curvearrowright \beta G}(n) \leq \text{Føl}_{(G;S)\curvearrowright \beta G}(n)$, while Proposition 4.4 gives $\text{Føl}_{(H;\Omega)\curvearrowright \beta H}(n) \leq \text{Føl}_{(H;\Omega)\curvearrowright \beta G}(n)$. □

In the same spirit we have

Proposition 4.6. *Let G and H be groups with Property A. Then*

$$\text{Føl}_{G\times H} \leq \text{Føl}_G \, \text{Føl}_H \, .$$

Proof. By Propositions 3.10 and 3.5 we have $\text{Føl}_{G\times H}(n) \leq \text{Føl}_{G\times H\curvearrowright \beta G\times\beta H}(n) \leq \text{Føl}_G \, \text{Føl}_H(n)$. □

The next property of the isoperimetric profiles is the estimate for quotients by amenable, normal subgroups. It is inspired by a result in [37].

Proposition 4.7. *Let G be a finitely generated group and H a finitely generated, normal, amenable subgroup of G. Let $\pi : G \to G/H$ be the quotient map. Then*

$$\text{Føl}_{(G/H;\,\pi(S))}(n) \leq \text{Føl}_{(G;S)}(n)$$

for every $n \in \mathbb{N}$.

For the proof we need an additional fact. The argument relies on an averaging procedure similar to the one used earlier in this article. The reader is referred to [37, Proposition 3] for a detailed proof.

Proposition 4.8 ([37])**.** *Let G be a group with Property A and let H be an amenable subgroup of G. Then for every $\varepsilon > 0$ the map in Definition 3.1 for the action on $\ell_\infty(G)$ can be realized by a map $\xi : G \to \ell_\infty(G)$ such that each coefficient ξ_g is invariant under the action of H, i.e.,*

$$h * \xi_g = \xi_g \tag{5}$$

for every $h \in H$ and $g \in G$.

Proof of Proposition 4.7. By Proposition 4.8, the function $\xi : G \to \ell_\infty(G)$ can be chosen to be constant on cosets of H. Since $\ell_\infty(G/H)$ is exactly the subalgebra of $\ell_\infty(G)$ consisting of functions in $\ell_\infty(G)$ which are constant on the cosets of H, we can view ξ as a map $\xi : G \to \ell_\infty(G/H)$. For each coset of H fix a unique representative $x \in G$ and define the associated coset by $[x]$. The set of these chosen representatives is denoted by T. We define the map $\eta : G/H \to \ell_\infty(G/H)$ by

$$\eta_{[x]} = \sum_{h\in H} \xi_{hx}.$$

To see that η is well defined let $[x] = [x']$ as elements of G/H. Then

$$\eta_{[x']} = \sum_{h\in H} \xi_{hx'} = \sum_{h\in H} \xi_{h\gamma x} = \eta_{[x]},$$

where $\gamma \in H$ is such that $\gamma x = x'$. Clearly η is finitely supported and we have

$$\begin{aligned} \sum_{[x]\in G/H} \eta_{[x]} &= \sum_{x\in T} \eta_{[x]} \\ &= \sum_{x\in T} \sum_{h\in H} \xi_{hx} \\ &= \sum_{g\in G} \xi_g \;=\; \mathbb{1}_G, \end{aligned}$$

and under our description of $\ell_\infty(G/H)$ we have $\mathbb{1}_G = \mathbb{1}_{G/H}$. Let now $\sigma = \pi(s) \in \pi(S)$. Then

$$\begin{aligned} \|\eta - \sigma\cdot\eta\|_{\ell_1(G/H;\ell_\infty(G/H))} &= \sum_{[x]\in G/H} \left| \eta_{[x]} - \sigma * \eta_{\sigma^{-1}[x]} \right| \\ &= \sum_{x\in T} \left| \sum_{h\in H} \xi_{hx} - \sum_{h\in H} s * \xi_{s^{-1}hx} \right| \\ &\leq \sum_{x\in T} \sum_{h\in H} \left| \xi_{hx} - \xi_{s^{-1}hx} \right| \\ &\leq \sup_{s\in S} \|\xi - s\cdot\xi\|_{\ell_1(G;\ell_\infty(G))} \;\leq\; \varepsilon. \end{aligned}$$

Since $\#\operatorname{supp}\eta \leq \#\operatorname{supp}\xi$, we are done. ◻

The next theorem shows that the isoperimetric profile has a certain large-scale invariance property.

Theorem 4.9. *Let G be a finitely generated group and H a finite index subgroup of G. Then*

$$\text{Føl}_G \;\simeq\; \text{Føl}_H\,.$$

Proof. By Proposition 4.5 we get the estimate ”$\geq$” and we only need to show the other estimate. For this note that the inclusion $i : H \to G$ of a finite index subgroup is a quasi-isometry and there exists a number $C > 0$ such that for every $g \in G$ there exists $h \in H$ such that $d(g,h) \leq C$. For each $g \in G$ choose exactly one $h = j(g)$ satisfying this condition, in addition such that

$$j(hg) = hj(g)$$

for every $g \in G$ and every $h \in H$. This can be done for instance by choosing exactly one representative γ in each coset of H in G, defining $j(\gamma) = e$ and taking translates of these under the H action.

Now given $\eta : H \to \ell_\infty(H)$ satisfying the conditions of Definition 3.1 for a given $\varepsilon > 0$ take $\widetilde{\eta}_h(h') = \eta_{h^{-1}h'}(h')$ and define $\widetilde{\xi} : G \to \ell_1(G)$ by setting

$$\widetilde{\xi}_g(g') = \widetilde{\eta}_{j(g)}(j(g)).$$

Finally we define

$$\xi_g(g') = \widetilde{\xi}_{gg'}(g')$$

for all $g, g' \in G$.

We now need to check the conditions of Definition 3.1. Let $N = j^{-1}(e)$. First it is easy to see that since for every $g' \in G$ we have

$$\sum_{g \in G} \widetilde{\xi}_g(g') = \sum_{g \in G} \eta_{j(g)}(j(g')) \geq 1,$$

we also obtain

$$\sum_{g \in G} \xi_g \geq \mathbb{1}_G.$$

Second, we need to check the ε-condition. Note that the condition

$$\|\eta - s \cdot h\|_{\ell_1(G;\ell_\infty(G))} \leq \varepsilon,$$

where s is a generator of H, is equivalent to the condition

$$\|\widetilde{\eta}_h - \widetilde{\eta}_{h'}\|_{\ell_1(H)} \leq \varepsilon$$

whenever $d(h,h') \leq 1$. In this setting, we obtain

$$\|\widetilde{\xi}_g - \widetilde{\xi}_{g'}\|_{\ell_1(G)} = N\|\widetilde{\eta}_{j(g)} - \widetilde{\eta}_{j(g')}\|_{\ell_1(G)}.$$

Since $j : G \to H$ is a quasi-isometry, there exists a number C such that $d(g,g') \leq 1$ implies $d(j(g), j(g')) \leq C$ and we have $\|\widetilde{\xi}_g - \widetilde{\xi}_{g'}\|_{\ell_1(G)} \leq \varepsilon$. Reversing the first step we see that

$$\|\xi - s \cdot \xi\|_{\ell_1(G;\ell_\infty(G))} \leq N\varepsilon.$$

We normalize ξ and this preserves the above estimate. Finally we note that

$$\#\operatorname{supp} \xi \leq N \#\operatorname{supp} \eta.$$

This proves the theorem. □

Two groups are commensurable if they have isomorphic finite index subgroups. We have

Corollary 4.10. *Let G and H be commensurable. Then* $\mathrm{Føl}_G \simeq \mathrm{Føl}_H$.

Question 1.1 stated in the introduction is very natural here. For instance if the isoperimetric profile is a quasi-isometry invariant we would be able to use the Švarc-Milnor lemma to naturally define a generalized isoperimetric profile of a manifold with a cocompact action of a group with Property A (see also Section 6.1 for a general construction of such a profile). We believe the answer to Question 1.1 is positive, however we haven't found a suitable argument. The problem here is in the definition of the map j, which above could be chosen to be H-equivariant. However in the general case this is not necessarily so. In other words, using different definitions of Property A, one can phrase the above problem by saying that given a quasi-isometry $f : G \to H$ the induced product map $f \times f : G \times G \to H \times H$ does not necessarily take diagonals (that is sets of the form (g, gh) for a fixed $h \in G$) to diagonals, or even to bounded neighborhoods of diagonals.

The notion of a *lossless isoperimetric profile* of the group is defined in remark 4.18. However it is more appropriate to state now the following

Proposition 4.11. *Let C be the class of groups with Property A and a isoperimetric profile which is lossless with respect to the isodiametric profile* A. *If G and H are finitely generated groups belonging to C then*

$$\mathrm{F}\o\mathrm{l}_G \simeq \mathrm{F}\o\mathrm{l}_H\,.$$

This covers all groups for which the isodiametric profile has been computed so far, but we don't know if this holds in general.

4.2. **Isoperimetric profiles and asymptotic dimension.** In [34] an isodiametric profile A_G for groups with Property A was studied, together with a relation to type of asymptotic dimension. This estimate turns out to give sharp bounds on the asymptotics of the isoperimetric profile and our goal in this section is to use this principle again.

We recall the definitions. The isodiametric profile of a group with Property A is a function $\mathrm{A}_G : \mathbb{N} \to \mathbb{N}$ defined by

$$\mathrm{A}_G(n) = \inf_{\xi} \left(\inf \{S > 0 \ : \ \operatorname{supp} \xi \subseteq B(e, S)\}\right),$$

where $\xi : G \to \ell_1(G; \ell_\infty(G))$ satisfies the conditions of Definition 3.1 for $\varepsilon = \frac{1}{n}$.

We will skip the discussion of the isodiametric profile A_G itself and pass on directly to type of asymptotic dimension, recalling only that $\tau_{k,X} \geq A_X$ for any metric space with $\operatorname{asdim} X \leq k$ [34, Theorem 5.3] (see Definition 4.13 below for $\tau_{k,X}$). In all of our considerations τ can be replaced by A without any change. The reader is referred to [34] for a detailed study of the isodiametric profiles. A family $\mathcal{U}$ of subsets of a metric space is δ-bounded if $\operatorname{diam} U \leq \delta$ for every $U \in \mathcal{U}$. Two families $\mathcal{U}_1, \mathcal{U}_2$ are R-disjoint if $d(U_1, U_2) \geq R$ for any $U_1 \in \mathcal{U}_1$, $U_2 \in \mathcal{U}_2$.

Definition 4.12 ([22]). *We say that a metric space X has asymptotic dimension less than $k \in \mathbb{N}$, denoted* $\operatorname{asdim} X \leq k$, *if for every $R < \infty$ one can find a number $\delta < \infty$ and $k + 1$ R-disjoint families $\mathcal{U}_0, ..., \mathcal{U}_k$ of subsets of X such that*

$$X = \mathcal{U}_0 \cup ... \cup \mathcal{U}_k$$

and every $\mathcal{U}_i$ is δ-bounded

The importance of asymptotic dimension has become apparent after a result of Yu [47], who proved that the Novikov Conjecture holds for groups with finite asymptotic dimension. We refer the reader to [5, 6, 40] for more details on the notion of asymptotic dimension.

Definition 4.13. *Let X be a metric space satisfying* $\operatorname{asdim} X \leq k$. *Define the type function $\tau_{k,X} : \mathbb{N} \to \mathbb{N}$ in the following way: $\tau_{k,X}(n)$ is the smallest $\delta \in \mathbb{N}$ for which X can be covered by $k+1$ families $\mathcal{U}_0, ..., \mathcal{U}_k$ which are all n-disjoint and δ-bounded.*

We have the following corollary, regarding the type of asymptotic dimension. The proof is the same as in the [34, Theorem 6.1], we will sketch it and refer to [28, 34] for details.

Theorem 4.14. *Let G be a group satisfying* $\operatorname{asdim} G \leq k$. *Then there exists a constant $C > 0$ such that*

$$\mathrm{F\o l}_G \preceq \mathrm{Vol}_G \circ C\tau_{k,G}.$$

Proof. By Definition 4.13, for every $n \in \mathbb{N}$, X admits a cover by $k+1$, $\tau_{k,X}(n)$-bounded, n-disjoint families $\mathcal{U}_i$. Let $\mathcal{U}$ be a cover of X consisting of all the sets from all the families $\mathcal{U}_i$. There exists a partition of unity $\{\psi_V\}_{V\in\mathcal{U}}$ and a constant C_k depending on k such that (1) each ψ is Lipschitz with constant $2/n$; (2) $\sup \operatorname{diam}(\operatorname{supp}\psi) \leq C_k\tau_{k,X}(n)$; (3) for every $x \in X$ no more than $k+1$ of the values $\psi(x)$ are non-zero. For every ψ choose a unique point x_ψ in the set $\operatorname{supp}\psi$ and define for every $g \in G$

$$\xi^n_g(\gamma) = \psi_\gamma(\gamma g).$$

Then it is easy to check that $\|\xi - s\cdot\xi\|_{\ell_1(G;\ell_\infty(G))} \leq \frac{2}{n}C'_k$, where C'_k is another constant depending on k only and $\operatorname{supp}\ \xi^n_x \subseteq B(x, C_k\tau_{k,X}(C'_k n))$. □

Remark 4.15. The reader can immediately see that the above theorem should allow a generalization to profiles of general action, once type of asymptotic dimension is replaced by an appropriate notion. We will not pursue this here.

Finite asymptotic dimension for which τ is linear is known as *linearly controlled asymptotic dimension*, *asymptotic dimension with Higson property*, or in the case of discrete spaces as *finite Assouad-Nagata dimension*. See [12, 18, 32] for more details. The following is a straightforward consequence of Theorem 4.14 and a generalization of [34, Corollary 7.1].

Corollary 4.16. *Let G be a group with finite asymptotic dimension of linear type. Then* $\mathrm{F\o l}_G \simeq \mathrm{Vol}_G$. *In particular this conclusion holds for all non-amenable groups with finite asymptotic dimension of linear type.*

Proof. The upper bound is Theorem 4.14 and the lower bound follows from Theorem 3.15. □

Many authors have studied groups which embed quasi-isometrically into finite products of trees or hyperbolic spaces, see e.g [11] and the references therein. Such products have finite Assouad-Nagata dimension.

Corollary 4.17. *Let G be a group which embeds quasi-isometrically into a finite product of trees. Then* $\mathrm{Føl}_G \simeq \mathrm{Vol}_G$.

Remark 4.18. As mentioned earlier, the type of asymptotic dimension is a special case of the isodiametric profile A_G of a group with Property A, as introduced in [34]. We call the isoperimetric profile *lossless with respect to* φ if $\mathrm{Føl}_G \simeq \mathrm{Vol}_G \circ \varphi$. In all the above cases Føl is lossless with respect to τ and since both the isodiametric profile and volume growth are quasi-isometry invariants, the generalized isoperimetric profile is also invariant under quasi-isometries among groups with lossless isoperimetric profiles. Thus also question 1.2 from the introduction, whether isoperimetric profiles of all groups with Property A are lossless, is natural here. At present it is the case for all the known examples.

5. Explicit computation of isoperimetric profiles

In [34] it was already shown that type of asymptotic dimension can be used to determine exactly isoperimetric profiles of certain amenable groups. In this section we employ the relation to asymptotic dimension to compute exact asymptotics of some non-amenable groups. We repeatedly use the Milnor-Švarc lemma which says that a if a group G acts properly, cocompactly by isometries on a compact space X then G and X are quasi-isometric.

One of the basic facts about subexponential volume growth is that it implies amenability, thus non-amenable groups have exponential growth. Using type of asymptotic dimension we calculate isoperimetric profiles of the following groups, most of which are non-amenable.

5.0.1. *Hyperbolic groups.* Hyperbolic groups are the ones characterized by the δ-thin triangles condition (see e.g. [8]). Except when the group is elementary hyperbolic they are never amenable and some of them have Kazhdan's Property (T) (see [50]). However they have finite asymptotic dimension of linear type and we can see that for any hyperbolic group G we have

$$\mathrm{Føl}_G \simeq \mathrm{Vol}_G \simeq \begin{cases} n & \text{if } G \text{ is elementary} \\ \exp & \text{if } G \text{ is non-elementary} \end{cases}$$

See also the last section for additional remarks about uniform rates of hyperbolic groups.

5.0.2. *Baumslag-Solitar groups.* Recall that the Baumslag-Solitar groups are the one relator groups given by the presentation

$$B(m,n) = \left\langle a, b \mid ab^m a^{-1} = b^n \right\rangle$$

for $m, n \in \mathbb{Z}$. Such a group is amenable if and only if $|n| = 1$. It is well-known [39] (see also [34]) that $\mathrm{Føl}_{B(1,m)} \simeq \exp$. The Baumslag-Solitar groups act properly, cocompactly by isometries on a complex which can be described as a warped product of a tree and $\mathbb{R}$, where the metric on the product is warped in such a way that in the natural projection onto the tree the preimage of an edge is a horostrip of constant curvature $-\ln \frac{n}{m}$. The reader can easily verify that this warped product has finite asymptotic dimension of linear type. Baumslag-Solitar groups also have exponential growth. Thus for all $m, n \in \mathbb{Z}$ we have

$$\mathrm{F\o l}_{B(m,n)} \simeq \exp,$$

which generalizes the estimate previously known for the solvable Baumslag-Solitar groups. It is is also plausible that the various generalizations of Baumslag-Solitar groups [20, 43] have similar properties.

5.0.3. *Euclidean buildings of rank $n \geq 1$.* In [32] it was proved that any Euclidean building X of rank $n \geq 1$ has Assouad-Nagata dimension n. Consequently if G is a finitely generated group acting properly, cocompactly by isometries on a Euclidean building of rank at least 1 then

$$\mathrm{F\o l}_G \simeq \mathrm{Vol}_G .$$

5.0.4. *Hadamard manifolds.* A Hadamard manifold is simply-connected, complete manifold which has everywhere non-positive sectional curvature. In [32] it was proved that a homogeneous Hadamard manifold has finite Assouad-Nagata dimension. Thus if we let G be a group acting properly, cocompactly by isometries on a homogeneous Hadamard manifold then

$$\mathrm{F\o l}_G \simeq \mathrm{Vol}_G .$$

5.0.5. *Coxeter groups.* Recall that a Coxeter group $(G; S)$ is described by relations $s_i^2 = 1 = (s_i s_j)^{m_{ij}}$ where $m_{ij} \in \mathbb{Z}\backslash\{1\}$. In [29] (see also [17]) it was proved that every Coxeter group embeds quasi-isometrically into a finite product of trees. Thus, by Corollary 4.17 for any Coxeter group G we have

$$\mathrm{F\o l}_G \simeq \mathrm{Vol}_G .$$

5.1. **Group constructions.** There are also some group constructions for which we can compute the isoperimetric profile.

5.1.1. *Direct products.* For all the above groups we can give an explicit estimate for direct products. Namely if G and H belong to any class of groups mentioned earlier in this section, assuming additionally that at least one of the two groups has exponential growth, then

$$\mathrm{F\o l}_{G\times H} \simeq \exp .$$

This holds since by Proposition 4.6,

$$\exp \preceq \mathrm{F\o l}_{G\times H} \preceq \mathrm{F\o l}_G\, \mathrm{F\o l}_H \simeq \exp^2 \simeq \exp$$

5.1.2. *Extensions of groups with finite Assouad-Nagata dimension.* In [10] the authors prove a Hurewicz-type theorem for Assouad-Nagata dimension, which says that given a short exact sequence of finitely generated groups

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1$$

the group Γ has finite Assouad-Nagata dimension provided that both N and H have finite Assouad-Nagata dimension and N is undistorted in Γ. In view of this we have that with the above assumptions,

$$\mathrm{Føl}_G \simeq \mathrm{Vol}_G\,.$$

Remark 5.1. There are of course groups for which Føl is not exponential. Erschler [19] proved that for the wreath product $G \wr H$ of amenable group the isoperimetric profile satisfies $\mathrm{Føl}_{G \wr H} \simeq (\mathrm{Føl}_G)^{\mathrm{Føl}_H}$. A relevant question here is 1.3 from the introduction, namely whether this formula generalizes to all groups with Property A, or at least with finite asymptotic dimension. This would also show that the results of [9] are a manifestation of the same phenomenon as the examples of groups with finite asymptotic dimension but of non-linear type given in [34]. We remark that one requires an additional, mild assumption here, see [19] for details.

6. Final remarks and questions

6.1. **Generalization to metric spaces.** As mentioned earlier, for a group existence of an amenable action on some compact space is equivalent to Yu's Property A (see [36]). It is easy to generalize our construction to metric spaces which posses a certain structure of diagonals in the Cartesian product (in the group case such a diagonal is given by $\{g, gh\}_{g\in G}$ for a fixed $h \in G$) where the group action is replaced by partial translations. The details are left to the reader.

6.2. **Uniform growth rates.** Recall that the growth rate of a finitely generated group $(G; S)$ is the number $\omega_{(G;S)} = \lim_{n\to\infty}(\mathrm{Vol}(n))^{1/n}$ (the limit always exists due to submultiplicativity of the growth function) and the minimal growth rate $\omega_G = \inf_S \omega_{(G;S)}$. Similarly, we can define the growth rate of the isoperimetric profile of an action to be $\mathcal{F}_{(G;S)} = \liminf_{n\to\infty}(\mathrm{Føl}_{(G;S)\curvearrowright X}(n))^{1/n}$ (we don't know if the limit has to exist in this case), and the minimal growth rate $\mathcal{F}_G = \inf_S \mathcal{F}_{(G;S)}$. It follows from our results that there is a relation between the above minimal rates, namely

$$\omega_G \leq \mathcal{F}_G.$$

We say that G has uniform growth rate if $\omega_G > 1$, but note that this implies a uniform bound from below on the growth rate of the isoperimetric profile. In [3] the authors introduce a notion of uniform non-amenability, which is a strengthened version of non-amenability. They show that which are uniformly non-amenable have uniform exponential growth rate. This brings us to an interesting observation that uniformly non-amenable groups, such as free groups and non-elementary hyperbolic groups, have a uniform bound from below on the isoperimetric profiles.

6.3. **Final remarks and open questions.** There are several questions that arise in the context of the generalized isoperimetric profiles.

(1) It was proved by Guentner, Higson and Weinberger that linear groups have Property A. What are the isoperimetric profiles of linear groups? We conjecture that $\mathrm{Føl}_G \simeq \mathrm{Vol}_G$ for any linear group G.

(2) Guentner proved [25] that one-relator groups have Property A, and Matsnev [33] a stronger fact that they have finite asymptotic dimension. We conjecture that for any one-relator group $\mathrm{Føl}_G \simeq \mathrm{Vol}_G$.

(3) Is it possible to give a general formula for $\mathrm{Føl}_{G*H}$? Note that $\mathrm{Føl}_{\mathbb{Z}} \simeq n$ while $\mathrm{Føl}_{\mathbb{Z}*\mathbb{Z}} \simeq \exp$, but on the other hand $\mathrm{Føl}_{\mathbb{F}_2*\mathbb{F}_2} \simeq \mathrm{Føl}_{\mathbb{F}_4} \simeq \exp$.

(4) In [7] the authors introduced uniformly finite homology theory H_*^{uf} and characterized amenability of a discrete space X via non-vanishing of the group $H_0^{uf}(X)$. What is the counterpart for amenable actions?

It is plausible that the answer to the last question might shed some light on the problem of constructing explicit examples of groups without Property A.

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