Geometric induction in equivariant KK-theory

Piotr W. Nowak

Institute of Mathematics
Polish Academy of Sciences
Warsaw
pnowak@impan.pl

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Abstract

Given two quasi-isometric non-amenable groups G and H we construct an explicit map from H-equivariant KK-theory to G-equivariant KK-theory.

1 Statements and motivations

It is an interesting open question whether conjectures such as the Baum-Connes conjecture or the (strong) Novikov conjecture are geometrically invariant. More precisely, if G and H are quasi-isometric finitely generated groups and one of these conjectures holds for G, does it also hold for H. The purpose of this article is to take a step towards resolving this issue.

The first difficulty that arises is that the above conjecture, as formulated in terms of properties of the Baum-Connes assembly map, for two different groups on the left hand side involve the equivariant K-homology or equivariant KK-theory in the case of coefficients,

Theorem 1. *Let G and H be quasi-isometric non-amenable groups. Then there is a map in equivariant KK-theory,*

$$KK^H(A,B) \to KK^G(\mathscr{R}_X A, \mathscr{R}_X B).$$

This map is natural in the first coordinate.

In the above theorem, given an H-C^*-algebra A, by $\mathscr{R}_X A$ we denote the algebra of regulated functions from X to A, where X is a certain compact topological space (see Section 3).

The above map is particular instance of a map in KK-theory of cross-product groupoids by a homomorphism of groupoids. The construction itself is related closely to the previous constructions by Shalom [5], Sauer [4] and more recently X. Li [3]. We hope that they can be used towards proving certain forms of geometric invariance of the Baum-Connes, the Dirac-dual Dirac method, or the Novikov conjecture.

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2 Couplings and the associated cocycles

Consider two finitely generated groups, G and H. Assume that each of these groups is equipped with a finite generating set and the corresponding word length metric. G and H are then said to be quasi-isometric if there exists a map $f : G \to H$ and constants $C, L \geq 0$ such that

$$\frac{1}{L} d_G(g, g') - C \leq d_H(f(g), f(g')) \leq L d_G(g, g') + C,$$

for all $g, g' \in G$.

If the groups above are non-amenable then it was shown by Whyte that they are, in fact, bi-Lipschitz equivalent. We recall the following

Theorem 2 ([6])**.** *Let G and H be finitely generated groups and let $f : G \to H$ be a quasi-isometry. Then there exists a bijective quasi-isometry (i.e., a bi-Lipschitz equivalence) that is close to f. In particular, G and H are bi-Lipschitz equivalent.*

Gromov in [1] gave a dynamical characterization of quasi-isometry, that was later refined by Shalom [5] and Sauer [4].

Theorem 3 (Gromov [1, 0.2.C'$_2$]; Shalom [5, Theorem 2.1.2]; Sauer [4, Theorem 2.2])**.** *The groups G and H are quasi-isometric if and only if there exists a locally compact topological space Ω with commuting actions of G and H by homeomorphisms, that are both proper and cocompact.*

Moreover, we can choose a compact open fundamental domain $X \subseteq \Omega$, common for both actions, if and only if the two groups are bi-Lipschitz equivalent.

Sketch of the construction. Assume that G, H are bi-Lipschitz equivalent with a constant $L > 0$. Define Ω to be the space of all bijective maps $f : G \to H$, equipped with the pointwise convergence topology. The actions of G and H on Ω are given by pre-composing and post-composing with actions of G and H on themselves. Define the fundamental domain X to be the set of those maps $f \in \Omega$ that satisfy $f(e_G) = f(e_H)$. Then X is a common compact-open fundamental domain. □

Let Ω be a coupling of two finitely generated group G and H, as above. Let $\alpha : G \times X \to H$ and $\beta : X \times H \to G$ be the associated cocycles, defined by the formulas

$$xh \in \beta(x,h)X,$$
$$gx \in X\alpha(g,x).$$

In particular, $\alpha(e_G, x) = e_H$ for all $x \in X$. The cocycle α induces an action of G on X by the formula

$$g \cdot x = gx\alpha(g,x)^{-1}.$$

Similarly, we obtain an action of H on Y by

$$y \cdot h = \beta(y,h)^{-1}yh.$$

Assume now that there is a common compact-open fundamental domain $X \subseteq \Omega$ for both actions. This implies the formulas

$$\beta(x,\alpha(g,x)) = g^{-1}, \tag{1}$$
$$\alpha(\beta(x,h),x) = h. \tag{2}$$

for all $x \in X$. In particular for every $x \in X$ the restrictions

$$g \mapsto \alpha(g^{-1},x),$$

$$h \mapsto \beta(x,h),$$

are bijections and inverses of each other. By the definition of the actions on X we have

$$x \cdot \alpha(g,x) = \beta(x,\alpha(g,x))^{-1}x\alpha(g,x),$$

$$\beta(x,h)^{-1} \cdot x = \beta^{-1}(x,h)x\alpha(\beta(x,h),x),$$

which together with (1) and (2) yields

$$x \cdot \alpha(g,x) = g^{-1} \cdot x \tag{3}$$
$$\beta(x,h)^{-1} \cdot x = x \cdot h. \tag{4}$$

for every $x \in X$, $g \in G$. In other words, α and β allow to switch between the action of G and H on X. We also note the following

Lemma 4. *There exist constants $K, L > 0$ such that the inequalities*

$$|\alpha(g,x)| \le K|g|,$$

$$|\beta(x,h)| \le L|h|$$

hold for every $x \in X$ (here $|\cdot|$ denotes the word length).

The lemma can be easily proved using just the sketch of the construction of a topological coupling. In particular, once we fix $g \in G$, then

$$\alpha(g,\cdot) : X \to H,$$

takes only finitely values. Moreover,

$$|\beta(x,h)| \le L|h| = L|\alpha(\beta(x,h),x)|,$$

and since β is a bijection,

$$\frac{1}{L}|g| \le |\alpha(g,x)|,$$

for every $g \in G$.

The following general fact about couplings, that will be of use later.

Lemma 5. *Let G and H be quasi-isometric with the associated coupling as above. For each $g \in G$, $h \in H$ the set*

$$U_{g|h} = \{x \in X \ : \ \alpha(g,x) = h\}$$

is both open and closed.

Proof. Since the fundamental domain X is closed and open and the actions are continuous, the set is closed and open. □

Our standing assumption from now on is that we choose coupling given by the construction above.

3 The space of regulated functions

Consider a compact topological space X. The spaces we consider will, in general, have many proper subsets that are both closed and open. Let E be a Banach space. Consider now the space of functions $f: X \to E$ which are cut-and-paste continuous: for every $e \in E$, the set $f^{-1}(e)$ is both open and closed in X. The space of such functions was consider by Sauer in [4] and denoted $\mathscr{F}(X;E)$.

In particular, such functions are continuous and they form an algebra. Within this space we consider a certain natural subspace. By a step function we mean a function $f: X \to E$ whose set of values $\{f(x)\}_{x \in X} \subseteq E$ is finite. A cut-and-paste continuous step function is then of the form

$$f = \sum_{i=1}^{k} 1_{X_k},$$

where the X_k are open and closed pairwise disjoint subsets of X satisfying $\bigcup_{i=1}^{k} X_k = X$. The space of such functions will be denoted $\mathscr{S}_X E$.

Definition 6. *The space of regulated functions, denoted by $\mathscr{R}_X E$, is defined as the closure of the space of cut-and-paste continuous step functions inside $C(X;E)$, the space of continuous functions $X \to E$.*

The following properties are easy to verify.

Lemma 7. *Let $\mathscr{E}$ be a Banach space, X be a topological space and A be a C^*-algebra. Then*

1. *$\mathscr{R}_X A$ is a C^*-algebra with pointwise mulitplication. It is commutative if and only if A is.*
2. *If $\mathscr{E}$ is a right (left) Hilbert C^*-module over A then $\mathscr{R}_X \mathscr{E}$ is a right (left) Hilbert C^*-module over $\mathscr{R}_X A$, with the inner product given by*

$$\langle f, f' \rangle(x) = \langle f(x), f'(x) \rangle_A.$$

Let now X be a compact topological space arising from the quasi-isometry coupling. Assume additionally, that A is an H-Banach algebra (C^*-algebra). Then the algebra $\mathscr{R}_X A$ is a G-Banach (C^*-algebra) with the action of G given by

$$(g \cdot f)(x) = \alpha(g^{-1}, x)^{-1} f(g^{-1} \cdot x).$$

Assume now that we have an H-equivariant map

$$\zeta : A \to A',$$

between two H-Banach algebras (C^*-algebras). We have the obvious map

$$\zeta_* : \mathscr{R}_X A \to \mathscr{R}_X A'$$

induced by the composition $X \to A \to A'$.

Lemma 8. *The map ζ is G-equivariant.*

Proof.

$$\begin{aligned}(g\cdot(\zeta_* f))(x) &= \alpha(g^{-1},x)^{-1}(\zeta(f(g\cdot x)))\\ &= \zeta_*(\alpha(g^{-1},x)^{-1}f(g\cdot x))\\ &= (\zeta_*(gf))(x).\end{aligned}$$

□

It is clear that if $\zeta = \mathrm{Id}_A$ then $\zeta_* = \mathrm{Id}_{\mathscr{R}_X A}$, and that $(\zeta\circ\zeta')_* = \zeta_*\circ\zeta'_*$. Therefore, we have

Corollary 9. *Given X as above, $\mathscr{R}_X$ is a functor from the category of H-C^*-algebras to the category of G-C^*-algebras.*

We also record the following fact, which is a direct consequence of the Gelfand representation theorem.

Lemma 10. *Let $A = C_0(Y)$ where Y is equipped with a proper G-action. Then $\mathscr{R}_X A$ is a proper G-C^*-algebra.*

4 KK-induction via Quasi-Isometries

In this section we construct the map specified in Theorem 1. The map is constructed on the level of KK-cycles, below we outline all the elements necessary to do this.

Let (U,π,F) (on $\mathscr{E}$, a Hilbert C^*-module over B) be a KK-cycle over (A,B), where A, B are H-C^*-algebras. Out of this data we will construct an equivariant KK-cycle over the pair $(\mathscr{R}_X A, \mathscr{R}_X B)$ of G-C^*-algebras. We want to induce a map

$$KK^H_*(A,B) \to KK^G_*\big(\mathscr{R}_X A, \mathscr{R}_X B\big).$$

We fix a coupling of G and H, as in the sketch of the proof Theorem 3. Let X denote the fundamental domain.

4.1 The Hilbert module

Recall that X is a fundamental domain for H.

Consider the space $\mathscr{R}_X\mathscr{E}$. This space is a Hilbert C^*-module over $\mathscr{R}_X B$, with an inner product $\langle\cdot,\cdot\rangle \to \mathscr{R}_X B$ defined by $\langle f, f'\rangle_{\mathrm{reg}}(x) = \langle f(x), f(x)\rangle_{\mathscr{E}}$, as in lemma 7. Assume now that $\mathscr{E} = \mathscr{E}_1 \oplus \mathscr{E}_2$. Then

$$\mathscr{R}_X\mathscr{E} = \mathscr{R}_X\mathscr{E}_1 \oplus \mathscr{R}_X\mathscr{E}_2$$

is also a graded C^*-Hilbert module. Indeed, given projections $P_i : \mathscr{E} \to \mathscr{E}_i$, $i = 1,2$, we have that

$$(P_i\xi)(x) = P_i(\xi(x)), i = 1,2,$$

is an adjointable projection onto $\mathscr{R}_X\mathscr{E}_i$. If $\mathscr{E}_i$ were orthogonal subspaces of $\mathscr{E}$ a Hilbert space then then we have that $\mathscr{R}_X\mathscr{E}_i$ are orthogonal subspaces of $\mathscr{R}_X\mathscr{E}$.

4.2 The representation of the algebra

Recall that A is an H-C^*-algebra. Consider the C^*-algebra $\mathscr{R}_X A$. The action of G on $\mathscr{R}_X A$ is defined by

$$(g\cdot\phi)(x) = \alpha(g^{-1}, x)^{-1}\phi(g^{-1}\cdot x).$$

Since $\alpha(g,\cdot) \in H$ takes only finitely many values in H for a fixed $g \in G$, by lemma 5 the action is continuous.

Note that if the Hilbert module $\mathscr{E}$ is graded and representation of A preserves this grading, then the representation of $\mathscr{R}_X A$ also preserves the corresponding grading of $\mathscr{R}_X\mathscr{E}$.

4.3 The unitary representation of G

We define a unitary[1] representation of G on $\mathscr{R}_X\mathscr{E}$ using the cocycle provided the measure equivalence relation. More precisely, define a unitary representation $\widetilde{U}$ of G on $\mathscr{R}_X\mathscr{E}$ as before,

$$\widetilde{U}_g f(x) = U_{\alpha(g^{-1},x)^{-1}} f(g^{-1}\cdot x),$$

for $f : X \to \mathscr{E}$, $f \in \mathscr{R}_X\mathscr{E}$.

Note that

$$x \mapsto U^{-1}_{\alpha(g^{-1},x)},$$

[1] In the sense of Hilbert C^*-modules

as a map from X to the unitary group of $\mathscr{E}$, is cut-and-paste continuous, by lemma 5. Thus $\widetilde{U}_g f \in \mathscr{R}_X E$, provided $f \in \mathscr{R}_X A$.

Again we note that if the representation U preserves the grading of $\mathscr{E}$ then the $\widetilde{U}$ preserves the grading of $\mathscr{E}$.

$\mathscr{R}_X A$ is naturally represented on $\mathscr{E}$ via the formula

$$\widetilde{\pi}_\phi f(x) = (\phi \cdot f)(x) = \phi(x) f(x).$$

Then we have

$$\begin{aligned}
\langle \widetilde{U}_g f, \widetilde{U}_g f' \rangle_{\text{reg}}(x) &= \langle \pi_{\alpha(g^{-1},x)^{-1}} f(g^{-1}\cdot x), \pi_{\alpha(g^{-1},x)^{-1}} f(g^{-1}\cdot x) \rangle_{\mathscr{E}} \\
&= \alpha(g^{-1},x)^{-1} \langle f(g^{-1}\cdot x), f(g^{-1}\cdot x) \rangle_{\mathscr{E}} \\
&= \alpha(g^{-1},x)^{-1} \langle f, f' \rangle_{\text{reg}}(g^{-1}\cdot x) \\
&= \big(g \cdot \langle f, f' \rangle\big)(x).
\end{aligned}$$

That is, the representation $\widetilde{U}$ is unitary in the sense of Hilbert C^*-modules.

The next condition we verify is the covariance of the representation. We have

$$\begin{aligned}
\big(\widetilde{U}_g \phi \widetilde{U}_{g^{-1}} f\big)(x) &= \widetilde{U}_g \phi \big(U_{\alpha(g,x)^{-1}} f(g\cdot x)\big) \\
&= \widetilde{U}_g \big(\phi(x) U_{\alpha(g,x)^{-1}} f(g\cdot x)\big) \\
&= U_{\alpha(g^{-1},x)^{-1}} \phi(g^{-1}\cdot x) U_{\alpha(g,g^{-1}\cdot x)^{-1}} f(g\cdot g^{-1}\cdot x) \\
&= U_{\alpha(g^{-1},x)^{-1}} \phi(g^{-1}\cdot x) U_{\alpha(g,g^{-1}\cdot x)^{-1}} f(x)
\end{aligned}$$

By the cocycle condition,

$$\alpha(g, g^{-1}\cdot x)\alpha(g^{-1}, x) = \alpha(gg^{-1}, x) = \alpha(e, x) = e.$$

Thus

$$\begin{aligned}
\big(\widetilde{U}_g \phi \widetilde{U}_{g^{-1}} f\big)(x) &= U_{\alpha(g^{-1},x)^{-1}} \phi(g^{-1}\cdot x) U_{\alpha(g,g^{-1}\cdot x)^{-1}} f(x) \\
&= \alpha(g^{-1},x)^{-1} \phi(g\cdot x) f(x) \\
&= \big((g\cdot\phi) f\big)(x)
\end{aligned}$$

In other words, the map $\mathscr{R}_X A \to \mathscr{L}(\mathscr{R}_X E)$ is covariant with respect to the representation $\widetilde{U}$.

4.4 The Fredholm operator

Define now a linear operator $\widetilde{F}$ on $\mathscr{R}_X\mathscr{E}$ by setting

$$(\widetilde{F}f)(x) = Ff(x).$$

Clearly, $\widetilde{F}$ is self-adjoint on $\mathscr{R}_X\mathscr{E}$ in the sense of Hilbert C^*-modules. Also, if F reverses the grading of $\mathscr{E}$, which means

$$F = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix},$$

then

$$\widetilde{F} = \begin{pmatrix} 0 & \widetilde{T} \\ \widetilde{T}^* & 0 \end{pmatrix}.$$

That is, $\widetilde{F}$ is also preserving the grading.

We have

$$(\widetilde{U}_g\widetilde{F}f)(x) = U_{\alpha(g^{-1},x)^{-1}}Ff(x) = FU_{\alpha(g^{-1},x)^{-1}}f(x) = (\widetilde{F}\widetilde{U}_g f)(x)$$

We claim that $(\widetilde{U},\widetilde{\pi},\widetilde{F})$ defined on the Hilbert module $\mathscr{R}_X\mathscr{E}$ is in fact a KK-cycle. We thus want to show the following

Lemma 11. *Let $R : X \to \mathscr{L}(\mathscr{E})$ be a function taking finitely many values, each of which is a finite rank operator. Then R, viewed as an operator on $\mathscr{E}$, is a finite rank operator on $\mathscr{R}_X\mathscr{E}$.*

Proof. A rank one operator has the canonical form

$$R_{\eta,\xi}v = \eta\langle v,\xi\rangle_A.$$

A finite rank operator is a linear combination of finitely many of such operators, i.e.,

$$Rv = \sum_i R_{\eta_i,\xi_i}v = \sum_i \eta_i\langle v,\xi_i\rangle_A.$$

We have $R : X \to \mathscr{L}(E)$ as above of the form

$$Rv = \sum 1_{S_j}R_jv,$$

where S_j are disjoint subsets of X, $\bigcup S_j = X$. Thus for each v we have

$$Rv = \sum_j 1_{S_j}\sum_i R_{\eta_i^j,\xi_i^j}v = \sum_j\sum_i R_{1_{S_j}\eta_i^j,1_{S_j}\xi_i^j}v$$

Define

$$\widetilde{\eta}_i = \sum_j 1_{S_j} \eta_i^j,$$

$$\widetilde{\xi}_i = \sum_j 1_{S_j} \xi_i^j.$$

Then

$$Rv = \sum_i \eta_i \langle v, \xi_i \rangle_{\mathscr{R}_X A}.$$

□

Lemma 12. *The operator* $\widetilde{\pi}_\phi \widetilde{F} - \widetilde{F}\widetilde{\pi}_\phi$ *is compact in the sense of Hilbert* C^*-*modules for every* $\phi \in \mathscr{R}_X A$.

Proof. We have for $f \in \mathscr{R}_X \mathscr{E}$,

$$\big((\widetilde{\phi}\widetilde{F} - \widetilde{F}\widetilde{\phi})f\big)(x) = \widetilde{\phi}(x)\widetilde{F}f(x) - \widetilde{F}\widetilde{\phi}(x)f(x).$$

By density, it suffices to prove the above statement for simple $\phi : X \to A$.

Let $\varepsilon > 0$. For each $x \in X$ there exists a finite rank operator $R(x) \in B(\mathscr{E})$ (in the sense of Hilbert C^*-modules) such that for almost every $x \in X$

$$\big\| \widetilde{\phi}(x)\widetilde{F} - \widetilde{F}\,\widetilde{\phi}(x) - R(x) \big\| \le \varepsilon.$$

Since ϕ and f are both simple functions of x into their respective co-domains, we can choose $R : X \to B(\mathscr{E})$ to also take finitely many values. As such, the operator R is a finite rank operator in the sense of Hilbert C^*-modules.

Thus

$$\| \widetilde{\phi}\widetilde{F} - \widetilde{F}\widetilde{\phi} - R \| \le \varepsilon$$

which means that $\widetilde{\phi}\widetilde{F} - \widetilde{F}\widetilde{\phi}$ is compact in the sense of Hilbert modules. □

The same proof works for the other expression to give

Lemma 13. *The operator* $\widetilde{\pi}_\phi(\widetilde{F}^2 - 1)$ *is compact in the sense of Hilbert* C^*-*modules for every* $\phi \in \mathscr{R}_X C_0(\Xi)$.

The last property is given by

Lemma 14. *We have* $[\widetilde{U}, \widetilde{F}]$ *is compact.*

Proof. If in the original cycle (U,π,F) we only assume that $[U_h,F]$ is compact, not zero, for every $h \in H$, then we consider

$$\begin{aligned}(\widetilde{U}_g\widetilde{F}f)(x)-(\widetilde{F}\widetilde{U}_g f)(x) &= FU_{\alpha(g^{-1},x)^{-1}}f(g^{-1}\cdot x)-U_{\alpha(g^{-1},x)^{-1}}Ff(g^{-1}\cdot x)\\ &= R_x f(x)+\varepsilon,\end{aligned}$$

where ε is an operator of sufficiently small norm. Since f is a simple function and $\alpha(g^{-1},x)$ takes finitely many values in H for a fixed $g \in G$, we obtain that the operator R is a finite rank operator in the sense of Hilbert modules and it is ε-close to the commutator above. □

4.5 Homotopies

A homotopy between cycles (U,π,F_0) and (U,π,F_1) is path (U_i,π_i,F_i), $i \in [0,1]$, defined on a Hilbert module B, such that the map

$$t \mapsto F_t,$$

is norm continuous, and $U_i = U$, $\pi_i = \pi$ for all $i \in [0,1]$. We have the following

Lemma 15. *Let (U_i,π_i,F_i), $i \in [0,1]$ be a homotopy as above. Then $(\widetilde{U}_i,\widetilde{\pi}_i,\widetilde{F}_i)$, $i \in [0,1]$ is a homotopy connecting $(\widetilde{U}_0,\widetilde{\pi}_0,\widetilde{F}_0)$ and $(\widetilde{U}_1,\widetilde{\pi}_1,\widetilde{F}_1)$.*

Proof. Since the original homotopy is defined on a Hilbert module $\mathscr{E}$, the induced path is defined on $\mathscr{R}_X\mathscr{E}$. Also the induced representations $\widetilde{U}_i$ of G and $\widetilde{\pi}_i$ of $\mathscr{R}_X A$ are pairwise equal in $i \in [0,1]$. It is also clear that with this setup, the assignment $t \mapsto \widetilde{F}_t$ is a norm continuous path $[0,1] \to \mathscr{L}(\mathscr{R}_X\mathscr{E})$. □

4.6 KK-induction via couplings

With the above we are ready to prove the main theorem.

Theorem 16. *Let G and H be finitely generated groups which are quasi-isometric. Let Ω be the coupling sketched in Theorem 3 and X be a fundamental domain for the action of G on Ω. The map*

$$(U,\pi,F) \mapsto \left(\widetilde{U},\widetilde{\pi},\widetilde{F}\right)$$

induces a map in equivariant KK-theory,

$$KK_*^H(A,B) \to KK_*^G\left(\mathscr{R}_X A,\mathscr{R}_X B\right).$$

The map is natural in the first coordinate.

Proof. We already know that the map sends cycles to cycles. Also, it is clear that it sends degenerate cycles to degenerate cycles.

Assume now that $c = (U, \pi, F)$ and $c' = (U', \pi', F')$ represent the same KK-theory class in $KK_*^H(A,B)$. That means that there are degenerate cycles κ, κ' such that

$$c \oplus \kappa \sim c' \oplus \kappa'.$$

We need to show that the images of c and c' still represent the same KK-theory class in $KK_*^G(\mathscr{R}_X A, \mathscr{R}_X B)$. To this end denote by h_t, $t \in [0,1]$, the homotopy connecting the two cycles above, by Lemma 15. We observe that

$$\widetilde{h_t}$$

is a homotopy connecting $\widetilde{c} \oplus \widetilde{\kappa}$ and $\widetilde{c'} \oplus \widetilde{\kappa'}$.

We want to show that the induction map is natural in the first coordinate. That is, given an H-map $\phi : A \to A'$ the induced diagram in KK-theory commutes.

$$\begin{array}{ccc}
KK^H(A',B) & \longrightarrow & KK^H(A,B) \\
\downarrow & & \downarrow \\
KK^G\left(\mathscr{R}_X A', \mathscr{R}_X B\right) & \longrightarrow & KK^G\left(\mathscr{R}_X A, \mathscr{R}_X B\right)
\end{array}$$

The induced map $\phi_* : \mathscr{R}_X A \to \mathscr{R}_X A'$. In this case the only thing that changes is the representation of the algebra: the cocycle (U, π, F) is mapped to the cocycles $(U, \pi \circ \phi_*, F)$. Since ϕ_* is equivariant, the above construction preserves all the required properties [it seems clear - we only need to check the representations agree, since the finite rank business passes through, but it would be good to write it out].

$\square$

Remark 17 (*Discussion of naturality in the second variable)***.** It is not clear that given an H-equivariant map $\eta : B \to B'$ the quasi-isometric induction described above is functorial in the second variable. Namely, that the induced diagram in KK-theory,

$$\begin{array}{ccc}
KK^H(A,B) & \longrightarrow & KK^H(A,B') \\
\downarrow & & \downarrow \\
KK^G\left(\mathscr{R}_X A, \mathscr{R}_X B\right) & \longrightarrow & KK^G\left(\mathscr{R}_X A, \mathscr{R}_X B'\right)
\end{array}$$

is commutative. For instance, at the level of Hilbert modules, going through the top horizontal map and the right vertical QI induction map the resulting KK-cycle is defined on the $\mathscr{R}_X B'$ Hilbert C^*-module $\mathscr{R}_X(\mathscr{E} \otimes_\eta B')$.

On the other hand, the KK-cycle defined by following the left vertical QI induction map and then the bottom horizontal map is defined on the $\mathscr{R}_X B'$ Hilbert C^*-module $(\mathscr{R}_X \mathscr{E}) \otimes_{\eta_*} \big(\mathscr{R}_X B'\big)$.

These Hilbert C^*-modules are in general different but closely related. Consider the algebraic tensor product $v \otimes f \in \mathscr{S}_X \mathscr{E} \otimes_{\eta^*} \mathscr{S}_X B'$ where $v \in \mathscr{S}_X \mathscr{E}$ and $f \in \mathscr{S}_X B'$ are both functions on X. Define $\Phi : X \times X \to \mathscr{E} \otimes B'$ by the formula

$$\Phi(x, y) = v(x) \otimes f(y).$$

If $v \otimes f \neq 0$ then there must exist $x, y \in X$ such that $v(x) \neq 0$ and $f(y) \neq 0$. Then $\Phi \neq 0$ as well since $\Phi(x, y) = v(x) \otimes f(y) \neq 0$. Therefore the map is injective and we have an injection

$$\mathscr{R}_X \mathscr{E} \otimes \mathscr{R}_X B' \to \mathscr{R}_{X \times X}(\mathscr{E} \otimes B').$$

Together with the inclusion of the diagonal $X \simeq \{(x, x)\}_{x \in X} \subseteq X \times X$ we obtain

$$\mathscr{R}_X \mathscr{E} \otimes \mathscr{R}_X B' \to \mathscr{R}_{X \times X}(\mathscr{E} \otimes B') \to \mathscr{R}_X(\mathscr{E} \otimes B').$$

Now for the other components of the KK-cycle, going through the top arrow we have

$$(U, \pi, F; \mathscr{E}) \mapsto (U \otimes 1, \pi \otimes 1, F \otimes 1; \mathscr{E} \otimes_\psi B') \mapsto (\widetilde{U \otimes 1}, \mathscr{R}_X(\pi \otimes 1), \mathscr{R}_X(F \otimes 1); \mathscr{R}_X(\mathscr{E} \otimes_\psi B')),$$

where $\mathscr{R}_X \cdot$ denotes extending the $\cdot$ for each $x \in X$. On the other hand the bottom arrow gives

$$(U, \pi, F; \mathscr{E}) \mapsto (\widetilde{U}, \mathscr{R}_X \pi, \mathscr{R}_X F; \mathscr{R}_X \mathscr{E}) \mapsto (\widetilde{U} \otimes 1, (\mathscr{R}_X \pi) \otimes 1, (\mathscr{R}_X F) \otimes 1; \mathscr{R}_X E \otimes \mathscr{R}_X B')$$

We check that since $(U \otimes 1)_h = U_h \otimes 1$, on $\mathscr{E} \otimes_\eta B'$ we have

$$\begin{aligned}
(\widetilde{U \otimes 1})(f \otimes f')(x) &= (U \otimes 1)_{\alpha(g^{-1},x)^{-1}}(f \otimes f')(g^{-1} \cdot x) \\
&= \big(U_{\alpha(g^{-1},x)^{-1}} f(g^{-1} \cdot x)\big) \otimes \big(f'(g^{-1} \cdot x)\big) \\
&= \big(\widetilde{U}_g f(x)\big) \otimes \big(f'(g^{-1} \cdot x)\big) \\
&= \big(\widetilde{U}_g f(x)\big) \otimes \big(\rho_g f'(x)\big) \\
&= (\widetilde{U} \otimes \rho)_g (f \otimes f')(x),
\end{aligned}$$

where ρ is the representation of G on $\mathscr{R}_X A$ given by the action of G on X. It is possible that under certain conditions one could use the Fell absorption principle to show that the above two cycles in fact represent the same class in KK-theory.

5 Maps between crossed products

5.1 Crossed product rings and quasi-isometric induction

As before, let G, H be finitely generated groups and let A be an HC^*-algebra. We will now define a map between two objects: the crossed product of H and an algebra A and the crossed product of the space $\mathscr{S}_X A$ by G.

Let $f \in A \rtimes_{alg} H$, that is, f is a finitely supported function $f : H \to A$. Define $\widetilde{f} \in \mathscr{S}_X A \rtimes_{alg} G$,

$$\widetilde{f} : G \to \mathscr{S}_X A,$$

by setting

$$(\zeta_{(H:G)} f)_g(x) = \widetilde{f}_g(x) = f_{\alpha(g^{-1},x)^{-1}}.$$

We have

$$\sup_{x \in X} |\alpha(g,x)| \leq \infty,$$

for each $g \in G$. Thus for a fixed $g \in G$ we obtain that $\alpha(g,x)$ takes finitely many values so that $\widetilde{f}_g \in \mathscr{S}_X A$. Additionally, the fact that $\widetilde{f}_g$ is in fact a continuous step function follows from lemma 5.

Also, if we additionally assume that for each $g \in G$ there are $R, R' > 0$ such that

$$|g| \geq R \implies |\alpha(g,x)| \geq R',$$

almost everywhere in $x \in X$, then $\widetilde{f}$ has finite support and can be viewed as an element of $\mathscr{R}_X A \rtimes_{alg} G$.

Proposition 18. *The above map $\zeta_{(H:G)} : A \rtimes_{alg} H \to \mathscr{S}_X A \rtimes G$, $f \mapsto \widetilde{f}$ is a unital *-homomorphism.*

Proof. Denote by $\bullet$ the multiplication in the crossed product. We have

$$\begin{aligned}
(\widetilde{f} \bullet \widetilde{f}')_g(x) &= \sum_{\gamma \in G} \widetilde{f}_\gamma(x) \left(\gamma \cdot (\widetilde{f}')_{\gamma^{-1}g}(x)\right) \\
&= \sum_{\gamma \in G} \widetilde{f}_\gamma(x) \left(\alpha(\gamma^{-1},x)^{-1} \cdot \widetilde{f}'_{\gamma^{-1}g}(\gamma^{-1} \cdot x)\right) \\
&= \sum_{\gamma \in G} f_{\alpha(\gamma^{-1},x)^{-1}} \left(\alpha(\gamma^{-1},x)^{-1} \cdot f'_{\alpha((\gamma^{-1}g)^{-1},\gamma^{-1}\cdot x)^{-1}}\right) \\
&= \sum_{\gamma \in G} f_{\alpha(\gamma^{-1},x)^{-1}} \left(\alpha(\gamma^{-1},x)^{-1} \cdot f'_{\alpha(g^{-1}\gamma,\gamma^{-1}\cdot x)^{-1}}\right).
\end{aligned}$$

By the cocycle property of α we have

$$\alpha(g^{-1}\gamma, \gamma^{-1} \cdot x)\alpha(\gamma^{-1}, x) = \alpha(g^{-1}, x),$$

we continue the previous estimate to get

$$(\widetilde{f}\bullet\widetilde{f}')_g(x) = \sum_{\gamma\in G} f_{\alpha(\gamma^{-1},x)^{-1}}\left(\alpha(\gamma^{-1},x)^{-1}\cdot f'_{\alpha(\gamma^{-1},x)\alpha(g^{-1},x)^{-1}}\right).$$

If $\alpha(\gamma^{-1},x): G\to H$ is a bijection for every $x\in X$ then we obtain

$$(\widetilde{f}\bullet\widetilde{f}')_g(x) = (f\bullet f')_{\alpha(g^{-1},x)^{-1}}$$

As for the involution, we have

$$\begin{aligned}(\widetilde{f})^*_g(x) &= \overline{\widetilde{f}_{g^{-1}}(g^{-1}\cdot x)}\\ &= \overline{f_{\alpha(g,g^{-1}\cdot x)^{-1}}}\\ &= \overline{f_{\alpha(g^{-1},x)}}\\ &= (f^*)_{\alpha(g^{-1},x)^{-1}}\\ &= \widetilde{(f^*)}_g(x).\end{aligned}$$

□

The main question here is to what completions of $A\rtimes_{alg}H$ does the above map extend. It is not clear that it does so for the reduced one.

Remark 19. *Note that in the above we use properties of a coupling coming from a bi-Lipschitz equivalence.*

6 Quasi-isometric induction and the Baum-Connes assembly map

Let Ξ be a locally compact space with a proper action of H. Then $\mathscr{R}_X C_0(\Xi)$ is a proper algebra. Combining the above maps with the Baum-Connes assembly map we have the following diagrams ($i = 0,1$). It is not clear whether they are commutative.

$$\begin{array}{ccc} K_i^H(C_0(\Xi)) & \xrightarrow{\mu_i^{(H;\mathbb{C})}} & K_i(C^*_\square H) \\ \Big\downarrow \text{ME} & \searrow^{\zeta_{(H:G)}\circ\mu_i^{(H;\mathbb{C})}} & \Big\downarrow \zeta_{(H:G)} \\ KK_i^G(\mathscr{R}_X C_0(\Xi),\mathscr{R}_X\mathbb{C}) & \xrightarrow[\mu_i^{(G;\mathscr{R}_X\mathbb{C})}]{} & K_i(\mathscr{R}_X\mathbb{C}\rtimes G) \end{array}$$

Above, $C^*_{\square}H$ denotes an appropriately chosen $C*$-completion of the group ring, as it is not clear whether one can extend the map $\zeta_{(H:G)}$ from the group ring to e.g. the reduced group C^*-algebra.

One hope is that the above maps that allow to switch from H-equivariant theory to G-equivariant theory might allow to make some progress on the question whether the Baum-Connes conjecture or the strong Novikov conjecture are geometric.

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