CONTROLLED COARSE HOMOLOGY AND ISOPERIMETRIC INEQUALITIES

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ABSTRACT. We study a coarse homology theory with prescribed growth conditions. For a finitely generated group G with the word length metric this homology theory turns out to be related to amenability of G. We characterize vanishing of a certain fundamental class in our homology in terms of an isoperimetric inequality on G and show that on any group at most linear control is needed for this class to vanish. The latter is a homological version of the classical Burnside problem for infinite groups, with a positive solution. As applications we characterize existence of primitives of the volume form with prescribed growth and show that coarse homology classes obstruct weighted Poincaré inequalities.

1. INTRODUCTION

Isoperimetric inequalities are a fundamental tool in analysis and differential geometry. Such inequalities, including Sobolev and Poincaré inequalities and their numerous generalizations, have a large number of applications in various settings. In this paper we are interested in a discrete isoperimetric inequality studied by Żuk [29] and later by Erschler [9]. It is of the form

$$\#A \leq C \sum_{x \in \partial A} f(d(x, x_0))$$

for a fixed, non-decreasing real function f, a fixed point x_0 and a constant $C > 0$. The purpose of our work is to explore the connection between the inequality and one of the fundamental large-scale invariants, coarse homology.

Coarse homology and cohomology were first introduced by Roe in [19] for the purposes of index theory and allowed to formulate the coarse index on open manifold using assembly maps from coarse K-homology to the K-theory of appropriate C^*-algebras (see also [4, 20, 21]). This approach proved to be very successful in attacking various problems in geometry and topology of manifold such as the Novikov conjecture, positive scalar curvature problem or the zero-in-the-spectrum conjecture. Block and Weinberger [3] introduced and studied a uniformly finite homology theory, where they considered only those chains in Roe's coarse homology whose coefficients are bounded. This homology theory turned out to have many applications since vanishing of the 0-dimensional uniformly finite homology group characterizes amenability. Using this fact Block and Weinberger related it to the existence of aperiodic tilings and positive scalar curvature metrics on complete

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Riemannian manifolds [3]. Later Whyte used it to show the existence of bijective quasi-isometries (i.e. bilipschitz equivalences) between non-amenable groups [27]. Other applications can be found in e.g. [1, 2, 28].

The homology theory we study here is a controlled homology theory for spaces with bounded geometry, where the upper bound on the chains' growth type is specified and represented by a fixed, non-decreasing function f. The case when f is constant gives the uniformly finite homology of Block and Weinberger mentioned above, but our main interest is in the case when the chains are unbounded (see Section 2 for a precise definition). Roe's coarse homology is also defined using unbounded chains, but without any control on the coefficients' growth (it is a coarsened version of the locally finite homology). Our homology is a quasi-isometry invariant and contains information about the large-scale structure of a metric space.

For the most part, we restrict our attention to the case of finitely generated groups and we are mainly interested in vanishing of the fundamental class $[\Gamma] = \sum_{x\in\Gamma}[x]$ in the 0-th homology group $H_0^f(\Gamma)$. The main theme of our work is that the vanishing of this class describes "how amenable" a group is, through the isoperimetric inequality above. In the light of this philosophy and [29] one can expect that on any group killing the fundamental class should require a 1-chain with at most linear growth. Our first result confirms this with a direct construction of a linearly growing 1-chain whose boundary is the fundamental class. This fact restricts the class of growth types that are potentially interesting to those with growth between constant and linear. As we explain at the end of Section 3, the theorem can be viewed as a weaker, homological version of the classical Burnside problem for infinite groups with a positive solution. To the best of our knowledge this is the strongest result in this direction which is true for all infinite, finitely generated groups.

The result in section 4 states that the isoperimetric inequality holds on a given space if and only if the fundamental class vanishes in the corresponding coarse homology group and it is the second main result of the paper. This characterization generalizes the result of [3] which arises as the case of bounded control and also gives a homological perspective on Żuk's result [29]. Moreover, using the results of Erschler [9] we obtain explicit examples of groups for which the fundamental class bounds 1-chains growing much slower than linearly.

In section 5 we discuss explicit examples of amenable groups for which we can compute explicitly the control function f, for which the fundamental class vanishes in $H_0^f(\Gamma)$. The estimates are obtained using invariants such as isodiametric and isoperimetric profiles. These invariants are well-known and widely studied (see e.g. [9, 14, 17, 18, 22, 23]) and each of them can be used to estimate the growth of a 1-chain which bounds the fundamental class.

Our theory has some interesting applications to the problem of finding a primitive of a differential form on a universal cover of a compact manifold with prescribed growth. The question of finding such primitives was studied by Sullivan [24], Gromov [10] and Brooks [6] in a setting which relates to that of [3], and later in [25, 29]. Our results give exact estimates on this growth in the case when the group in question is amenable. Namely, we can use all of the above invariants as large-scale obstructions to finding such primitives. For instance, if Γ is an amenable group which has finite asymptotic dimension of linear type then the volume form on the universal cover of a compact manifold M with the fundamental group Γ cannot have a primitive of growth slower than linear. Our methods in particular give a different proof of a theorem of Sikorav characterizing the growth of primitives of differential forms [25].

Another connection is to weighted Poincaré inequalities studied by Li and Wang [13]. These inequalities are intended as a weakening of a lower bound on the positive spectrum of the Laplacian and were used in [13] to prove various rigidity results for open manifolds. It turns out that our isoperimetric inequalities and weighted Poincaré inequalities are closely related. We use this fact to show that the vanishing of the fundamental class is an obstructions to Poincaré inequalities with certain weights, which do not decay fast enough.

In a future paper we will use the theory presented here together with surgery theory to study positive scalar curvature on open manifolds, Pontrjagin classes and distortion of diffeomorphisms.

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2. Coarse homology with growth conditions

We will be considering metric spaces which are (uniformly) discrete in the sense that there exists a constant $C > 0$ such that $d(x, y) \geq C$ for any distinct points x, y in the given space. Additionally we assume that all our spaces have bounded geometry and are quasi-geodesic. A discrete metric space X has bounded geometry if for every $r > 0$ there exists a constant $N_r > 0$ such that $\#B(x, r) \leq N_r$ for every $x \in X$. X is said to be quasi-geodesic if for every $x, y \in X$ there exists a sequence of points $\{x = x_0, x_1, \ldots, x_{n-1}, x_n = y\}$ such that $d(x_i, x_{i+1}) \leq 1$ and $n \leq d(x, y)$

(this a slightly stronger definition than in the literature, but essentially equivalent). Examples of such spaces are finitely generated groups with word length metrics.

2.1. **Chain complex, homology and large-scale invariance.** Let us first introduce some notation. Let X be a discrete metric space with bounded geometry and let $e \in X$ be fixed. For $\overline{x} = (x_1, \dots, x_n), \overline{y} = (y_1, \dots, y_n) \in X^n$ we define the distance

$$d(\overline{x}, \overline{y}) = \max_i d(x_i, y_i),$$

and the length of $\overline{x}$ as $|\overline{x}| = d(\overline{x}, \overline{e})$ where $\overline{e} = (e, \dots, e)$. Let $f : [0, \infty) \to [0, \infty)$ be a non-decreasing function. For technical reasons we will assume that $f(0) = 1$ and that for every $K > 0$ there exists $L > 0$ such that

$$f(t + K) \leq L f(t) \tag{f_1}$$

for every $t > 0$. We will also assume that for every $K > 0$ there exists a constant $L > 0$ such that

$$f(Kt) \leq L f(t) \tag{f_2}$$

for every $t > 0$. Both conditions are mild and are satisfied by common sub-linear growth types such as powers and logarithms.

We shall define homology with coefficients in $\mathbb{R}$, but the definition of course makes sense for any normed ring, in particular $\mathbb{Z}$. We represent the chains in two ways: As a formal sum $c = \sum_{\overline{x} \in X^{n+1}} c_{\overline{x}} \overline{x}$, $c_{\overline{x}} \in \mathbb{R}$; or as a function $\psi : X^{n+1} \to \mathbb{R}$. In both cases, we think of $\overline{x} \in X^{n+1}$ as of simplices. In particular, we require the property $\psi(\overline{x}) = (-1)^{N(\sigma)} \psi(\sigma(\overline{x}))$, where σ is a permutation of the simplex $\overline{x} \in X^{n+1}$ and $N(\sigma)$ is the number of transpositions needed to obtain $\sigma(\overline{x})$ from $\overline{x}$. For 1-chains this simply reduces to $\psi(x, y) = -\psi(y, x)$.

Given a chain $c = \sum_{\overline{x} \in X^{n+1}} c_{\overline{x}} \overline{x}$ the *propagation* $\mathscr{P}(c)$ is the smallest number R such that $c_{\overline{x}} = 0$ whenever $d(\overline{x}, \Delta_{n+1}) \geq R$ for $\overline{x} \in X^{n+1}$, where Δ_{n+1} denotes the diagonal in X^{n+1}.

We now define the chain complex. Denote

$$C_n^f(X) = \left\{ c = \sum_{\overline{x} \in X^{n+1}} c_{\overline{x}} \overline{x} : \ \mathscr{P}(c) < \infty \text{ and } |c_{\overline{x}}| \leq K_c f(|\overline{x}|) \right\},$$

where K_c is a constant which depends on c and the coefficients $c_{\overline{x}}$ are real numbers. One can easily check using (f_1) that $C_n^f(X)$ is a linear space which does not depend on the choice of the base point.

We define a differential $\partial : C_n^f(X) \to C_{n-1}^f(X)$ in a standard way on simplices as

$$\partial([x_0, x_1, \dots, x_n]) = \sum_{i=0}^{n} (-1)^i [x_0, \dots, \hat{x}_i, \dots, x_n]$$

and by extending linearly. It is easy to check that (f_1) guarantees that the differential is well-defined. Thus we have a chain complex $\left\{ C_i^f(X), \partial \right\}$ and we denote its homology by $H_*^f(X)$. In particular, if $f \equiv \text{const}$, we obtain uniformly finite homology of Block and Weinberger [3] (with real coefficients), which we denote by $H_*^{\mathrm{uf}}(X)$ and if f is linear we will write $H_*^{\mathrm{lin}}(X)$.

We now turn to functorial properties of $H_*^f(X)$. First, we recall standard definitions.

Definition 2.1. *Let X and Y be metric spaces. A map $F : X \to Y$ is a* coarse equivalence *if there exist non-decreasing functions $\rho_-, \rho_+ : [0,\infty) \to [0,\infty)$ such that*

$$\rho_-(d_X(x,y)) \leq d_Y(F(x),F(y)) \leq \rho_+(d_X(x,y))$$

for all $x,y \in X$ and there exists a $C > 0$ such that for every $y \in Y$ there is an $x \in X$ satisfying $d_Y(F(x),y) \leq C$.

The map F is a quasi-isometry *if both ρ_- and ρ_+ can be chosen to be affine. It is a coarse (quasi-isometric)* embedding *if it is a coarse (quasi-isometric) equivalence with a subset of Y.*

Two coarse maps $F, F' : X \to Y$ are said to be close *if $d(F(x),F'(x)) \leq C$ for some constant $C > 0$ and every $x \in X$.*

Since we are only concerned with the asymptotic behavior, we can without loss of generality assume that ρ_- is strictly increasing and thus has an inverse. Let $F : X \to Y$ be a coarse embedding. Define

$$F_*\left(\sum c_x x\right) = \sum c_x F(x).$$

If $c \in C_i^f(X)$ then

$$|c_{F(x)}| \leq K f(|x|) \leq K f(\rho_-^{-1}(|F(x)|)),$$

since f is non-decreasing, where K depends on f and F only. Thus we obtain a map on the level of chains,

$$F_* : C_i^f(X) \to C_i^{f\circ\rho_-^{-1}}(Y).$$

In particular, if F is a quasi-isometric embedding, we have the induced map $F_* : C_i^f(X) \to C_i^f(Y)$. We denote also by F_* the induced map on homology,

$$F_* : H_i^f(X) \to H_i^{f\circ\rho_-^{-1}}(Y).$$

Proposition 2.2. *Let $F, F' : X \to Y$ be quasi-isometric embeddings which are close. Then $F_*, F'_* : C_i^f(X) \to C_i^f(Y)$ are chain homotopic.*

Proof. The proof follows the one in [3]. If F and F' are close then the map $\{F,F'\} : X_1 \times \{0,1\} \to Y$ is a coarse map, so we need to show that $i_0, i_1 : X \to X \times \{0,1\}$ are chain homotopic. Let $H : C_i^f(X) \to C_{i+1}^f(X \times \{0,1\})$ be defined as the linear extension of

$$H(x_0,\ldots,x_i) = \sum_{j=0}^{i} (-1)^j ((x_0,0),\ldots,(x_j,0),(x_j,1),\ldots,(x_i,1)).$$

Then $\partial H + H\partial = i_{1*} - i_{0*}$. □

Corollary 2.3. *H_*^f is a quasi-isometry invariant, i.e. for a fixed f and a quasi-isometry $F : X \to Y$ we have $H_n^f(X) \cong H_n^f(Y)$ for every $n \in \mathbb{N}$.*

2.2. **Vanishing of the fundamental class.** The phenomenon that we want to explore is that the vanishing of the fundamental class

$$[\Gamma] = \sum_{x \in \Gamma} [x]$$

in $H_0^f(\Gamma)$ for f with a certain growth type has geometric consequences. The most interesting case is when the space in question is a Cayley graph of a finitely generated group. Theorem 3.1 shows that in this case the fundamental class vanishes when the growth type is at least linear (see also [29]). On the other hand, if f is a constant function and $H_0^f(\Gamma) = H_0^{\mathrm{uf}}(\Gamma)$ then vanishing of the fundamental class $[\Gamma]$ in $H_0^{\mathrm{uf}}(\Gamma)$ is equivalent to non-amenability of Γ [3, Theorem 3.1]. Our philosophy is that if $H_0^{\mathrm{uf}}(\Gamma)$ is not zero (i.e. Γ is amenable), then the vanishing of the fundamental class in $H_0^f(\Gamma)$ for an unbounded f should quantify "how amenable" Γ is, in terms of the growth of f: the slower the growth of f, the "less amenable" the group Γ is.

In the uniformly finite homology vanishing of the fundamental class is equivalent to vanishing of the 0-th homology group. However, this is not expected to happen in general for the controlled homology. We record the following useful fact.

Lemma 2.4. *Let $c = \sum_{x \in X} c_x[x] \in C_0^f(X)$ be such that $c_x \geq C > 0$ for some C. If $[c] = 0$ in $H_0^f(X)$ then $[X] = 0$ in $H_0^f(X)$.*

Proof. First we will show that under the assumption there exists $c' = \sum c'_x[x]$ such that $c_x \in \mathbb{N} \setminus \{0\}$ and such that $c = \partial\psi$ where $\psi \in C_1^f(X; \mathbb{Z})$. Denote $N = \sup_{x \in X} \# B(x, \mathscr{P}(\phi))$. Given c and ϕ such that $\partial\phi = c$ take $\kappa > 0$ sufficiently large to guarantee $\kappa C - N \geq 1$. We have $\partial(\kappa\phi) = \kappa c$. Now define

$$\psi(x, y) = \begin{cases} \lceil \kappa\phi(x, y) \rceil & \text{if } \phi(x, y) \geq 0 \\ \lfloor \kappa\phi(x, y) \rfloor & \text{if } \phi(x, y) < 0 \end{cases}$$

where $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$ are the "ceiling" and "floor" functions respectively. Then $\psi \in C_1^f(X; \mathbb{Z})$ and

$$\begin{aligned} \partial\psi(x) &\geq \sum_{y \in X} \kappa\phi(y, x) - 1 \\ &\geq \partial\kappa\phi(x) - N \\ &\geq 1. \end{aligned}$$

Now using the technique from [3, Lemma 2.4], for every $x \in X$ we can construct a "tail" $t_x \in C_1^f(X)$ such that $\partial t_x = [x]$. To build t_x note that since $\partial\psi(x) \geq 1$ then there must exist $x_1 \in X$ such that the coefficient of $\psi(x, x_1) \geq 1$. Then, since $\psi(x_1, x) \leq -1$ and $\partial\psi(x_1) \geq 0$ there must exist x_2 such that $\psi(x_2, x_1) \geq 1$, and so on. Then $t_x = [x, x_1] + [x_1, x_2] + \dots$. We apply the same procedure to $\psi - t_x$ and continue inductively. Since we are only choosing simplices that appear in ψ we have $\sum_{x \in X} t_x \in C_1^f(X)$. By construction $\partial(\sum_x t_x) = 1_X$. $\square$

3. An explicit linear 1-chain

In this section we prove the first of the main results of this paper and explain why we are mainly interested in f at most linear. One picture conveyed in [3] for vanishing of the fundamental class in $H_0^{\mathrm{uf}}(\Gamma)$ was the one of an *infinite Ponzi scheme* or *tails* (as t_x constructed above). More precisely, the vanishing is equivalent to the existence of a collection of tails $t_x \in C_1^{\mathrm{uf}}(\Gamma)$ of the form $[x, x_1]+[x_1, x_2]+[x_2, x_3]+\dots$ for each point $x \in \Gamma$, so that $\partial t_x = [x]$, and such that for any chosen radius, the number of tails passing through any ball of that radius is uniformly bounded. For amenable groups this is impossible to arrange. However, for any finitely generated group, we can always arrange *some* kind of scheme, where the number of tails passing through a ball will be controlled by an unbounded function. Such escape routes to infinity are also known as *Eilenberg swindles*.

The following example explains why a linear control might be sufficient in general. On $\mathbb{Z}$, the 1-chain $\sum_{n\in\mathbb{Z}} n[n, n+1]$ has linear growth and its boundary is the fundamental class $[\mathbb{Z}] = \sum_{x\in\mathbb{Z}}[x]$. If Γ has a quasi–isometrically embedded cyclic subgroup then we can, roughly speaking, just take the above 1-chain on every coset to bound the fundamental class of $[\Gamma]$. In general however, as various solutions to the Burnside problem show, an infinite group does not have to have infinite cyclic subgroups at all. Nevertheless, an infinite finitely generated group still has a lot of infinite geodesics, which can play the role of the cyclic subgroup.

Theorem 3.1. *Let Γ be a finitely generated infinite group. Then $[\Gamma] = 0$ in $H_0^{\mathrm{lin}}(\Gamma)$.*

Proof. Take a group $\Gamma = \langle S \rangle$ with a finite symmetric generating set S. The notation we shall use is compatible with the one in [29], modulo the fact that we use the left-invariant metric induced from S, while in that paper the right-invariant metric is used. Let g_0 be a bi-infinite geodesic through the identity $e \in \Gamma$, let $\mathscr{G} = \{\gamma g_0 \mid \gamma \in \Gamma\}$ be a left-translation invariant set of parametrized geodesics (i.e. geodesics with a distinguished point). Let $G \subset \mathscr{G}$ be the set of all geodesics from $\mathscr{G}$ passing through e. We say that a subset H of $\mathscr{G}$ is measurable, if it is a subset of a set of the form $\delta_1 G \cup \dots \cup \delta_k G$ for some $\delta_1, \dots, \delta_k \in \Gamma$. Denote by $\mathscr{F}$ the set of measurable subsets of $\mathscr{G}$. Żuk's construction [29, Section 3.2], applied to a group with a left-invariant metric, produces a finitely additive measure φ on $\mathscr{F}$, which is left-invariant, and for which $\varphi(G) = 1$.

Let us remark that in general Żuk makes use of the invariant mean on $\mathbb{Z}$. However, if one can choose g_0 in such a way that $N = \#\{\gamma g_0 \mid \gamma \in g_0\} < \infty$ (now we consider the geodesics γg_0 as unparametrized), then the construction of such a measure greatly simplifies; see [29, Question 1]. Indeed, then one can use just unparametrized geodesics and set $\varphi(\{\gamma g_0\}) = \frac{1}{N}$.

Note that given any path $s \subset \Gamma$, we can think of it as a 1-chain in $C_1^{\mathrm{uf}}(\Gamma)$ with propagation 1: s is just a sequence of points (finite, infinite or bi-infinite), e.g. $(\dots, \gamma_{-1}, \gamma_0, \gamma_1, \dots)$, and we view s as the chain $\sum_{n\in\mathbb{Z}}[\gamma_n, \gamma_{n-1}] \in C_1^{\mathrm{uf}}(\Gamma)$.

On each $h \in \mathscr{G}$, let us distinguish another point $p(h) \in h$, which realizes the distance from e to the set h (if there are more such points, just choose one arbitrarily). Furthermore, for each parametrized geodesic $h \in \mathscr{G}$ containing δ, choose the subray $\vec{s}_\delta(h)$ of h which begins at δ and does not contain $p(h)$ (if $\delta = p(h)$, choose one of the rays arbitrarily). If h does not contain δ, just put $\vec{s}_\delta(h) = 0$.

For each $\delta \in \Gamma$, we would like to define a "spread tail" $t_\delta \in C_1^{\mathrm{uf}}(\Gamma; \mathbb{R})$ as

$$t_\delta = \sum_{h \in \mathcal{G}} \varphi(\{h\}) \vec{s}_\delta(h),$$

but if N is not finite, then $\varphi(\{h\}) = 0$. Instead, we define t_δ as follows: for each edge $(\gamma, \gamma s)$ ($s \in S$) in the Cayley graph of Γ, define the set of geodesics

$$A_\delta(\gamma, \gamma s) = \{h \in \mathcal{G} \mid \delta \in h \text{ and } (\gamma, \gamma s) \subset \vec{s}_\delta(h) \text{ preserving the direction}\}.$$

Note that $A_\delta(\gamma, \gamma s) \subset \delta G$ is measurable, with measure ≤ 1. Now define

$$t_\delta = \sum_{\gamma \in \Gamma, s \in S} \varphi(A_\delta(\gamma, \gamma s))[\gamma s, \gamma]$$

Obviously, the cycle t_δ has coefficients uniformly bounded by 1.

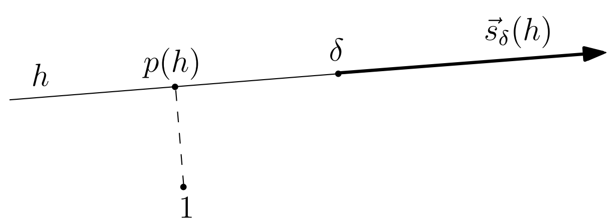

FIGURE 1. A tail

Lemma 3.2. $\partial t_\delta = [\delta]$

Proof. Consider any point $\gamma \in \Gamma \setminus \{\delta\}$, and note that there are only finitely many edges $(\gamma, \gamma s)$, $s \in S$ going out of the vertex γ. The coefficient of $[\gamma]$ in ∂t_δ is

$$\sum_{s \in S} \varphi(A_\delta(\gamma, \gamma s)) - \sum_{s \in S} \varphi(A_\delta(\gamma s, \gamma)) = \varphi\Big(\bigcup_{s \in S} A_\delta(\gamma, \gamma s)\Big) - \varphi\Big(\bigcup_{s \in S} A_\delta(\gamma s, \gamma)\Big)$$

Any ray $\vec{s}_\delta(h)$ going through γ contributes h to each of the sets $\bigcup_{s \in S} A_\delta(\gamma, \gamma s)$ and $\bigcup_{s \in S} A_\delta(\gamma s, \gamma)$. On the other hand, if the ray $\vec{s}_\delta(h)$ does not pass through γ, h is not contained in any of these sets. Hence, the expression in the above display is 0. Finally, looking at the coefficient of $[\delta]$ in ∂t_δ, observe that the rays $s_\delta(h)$ only go away from δ and all the geodesics $h \in \mathcal{G}$ passing through δ are used. Consequently, the coefficient is $\varphi(\bigcup_{s \in S} A_\delta(\delta, \delta s)) = \varphi(\delta G) = 1$. □

Now let $\psi = \sum_{\delta \in \Gamma} t_\delta$. Clearly ψ has propagation 1 and $\partial \psi = \sum_{\delta \in \Gamma} [\delta]$. The proof of the theorem is finished by the following Lemma. □

Lemma 3.3. $\psi \in C_1^{\mathrm{lin}}(\Gamma; \mathbb{R})$*, i.e.* ψ *has linear growth.*

Proof. The idea behind the argument is the following: take $\delta \in \Gamma$ and a geodesic $h \in \mathcal{G}$, which passes through δ. We need to count those points β on h, for which the chosen ray $\vec{s}_\beta(h)$ passes through δ. It is easy to see that these are precisely the points on h between $p(h)$ and δ. By the triangle inequality and the choice of $p(h)$, we have that the number $K(\delta, \gamma)$ of those points is at most $|p(\gamma)| + |\delta| \leq 2|\delta|$, where by our convention $|\gamma|$ denotes the length of the element γ. It follows that the coefficient of edges "attached" to δ is at most

$$\sum_{h \in \delta G} \varphi(\{h\}) K(\delta, \gamma) \leq \sum_{h \in \delta G} 2\varphi(\{h\})|\delta| = 2\varphi(\delta G)|\delta| = 2|\delta|.$$

Of course, the problem is again that all $\varphi(\{h\})$ can vanish and so we need to adjust the argument.

Pick any edge $(\gamma, \gamma s)$ in the Cayley graph of Γ and denote by $c_{[\gamma,\gamma s]}$ the coefficient of $[\gamma, \gamma s]$ in ψ. From the definition, we have $c_{[\gamma,\gamma s]} = \sum_{\delta \in \Gamma} \varphi(A_\delta(\gamma, \gamma s))$. The task is to prove that $c_{[\gamma,\gamma s]}$ depends linearly on $|\gamma|$.

We are going to further split each $A_\delta(\gamma, \gamma s)$. Let $P(\gamma, \gamma s)$ be the collection of all finite (non-parametrized) geodesic paths in Γ, which are subpaths $[p(h), \gamma]$ of some ray $\vec{s}_\delta(h)$ containing $(\gamma, \gamma s)$, $\delta \in \Gamma$, $h \in \mathcal{G}$. The collection $P(\gamma, \gamma s)$ is finite. Indeed, for any $\vec{s}_\delta(h)$ containing $(\gamma, \gamma s)$ we have that $|p(h)| \leq |\gamma|$ and the assertion follows from bounded geometry.

For each $\delta \in \Gamma$ and $a \in P(\gamma, \gamma s)$, we denote

$$\begin{aligned} B(a) &= \{h \in \mathcal{G} \mid \vec{s}_\delta(h) \text{ begins with } a\}, \\ B(\delta, a) &= \{h \in A_\delta(\gamma, \gamma s) \mid \vec{s}_\delta(h) \text{ begins with } a\}. \end{aligned}$$

All these sets are measurable, since they are subsets of γG. Furthermore $A_\delta(\gamma, \gamma s) = \bigsqcup_{a \in P(\gamma,\gamma s)} B(\delta, a)$. Note that $B(\delta, a)$ is either empty (when $\delta \notin a$), or equal to $B(a)$ (if $\delta \in a$). For a given $a \in P(\gamma, \gamma s)$, the number of δ for which $\delta \in a$ is bounded by $\mathrm{length}(a) = d(\gamma, p(h)) \leq |\gamma| + |p(h)| \leq 2|\gamma|$ (where h is arbitrary element of $B(a)$). Putting this information together, we obtain an estimate

$$\begin{aligned} c_{[\gamma,\gamma s]} &= \sum_{\delta \in \Gamma} \varphi(A_\delta(\gamma, \gamma s)) \\ &= \sum_{\delta \in \Gamma} \sum_{a \in P(\gamma,\gamma s)} \varphi(B(\delta, a)) \\ &= \sum_{a \in P(\gamma,\gamma s)} \sum_{\delta \in a} \varphi(B(a)) \\ &\leq 2|\gamma| \cdot \left(\sum_{a \in P(\gamma,\gamma s)} \varphi(B(a)) \right) \\ &\leq 2|\gamma| \varphi(\gamma G) \\ &= 2|\gamma|. \end{aligned}$$

For completeness, note that the sum over $\delta \in \Gamma$ in the above display has in fact finitely many non-zero terms. The last inequality uses the fact that $B(a)$'s are disjoint for different a's. □

Motivated by his results on a geometric version of the von Neumann conjecture [27] Whyte asked whether a geometric version of the Burnside problem problem has a positive solution. The geometric Burnside problem, as formulated in [27], asks whether every infinite group admits a free translation action of $\mathbb{Z}$, where a translation is understood as a bijective map which is close to the identity. When a group has a cyclic subgroup then clearly the left translation action by this subgroup is such an action.

Note that if the answer would be affirmative and, additionally, the orbits would be undistorted in the group G, then such undistorted geometric Burnside problem implies Theorem 3.1 by simply copying the 1-chain $\sum n[n, n+1]$ onto every orbit. This means that Theorem 3.1 in fact gives an affirmative answer to a weak, homological version of the Burnside problem. In other words, from the point of view of coarse homology, every infinite finitely generated group behaves as if it had an undistorted infinite cyclic subgroup. To the best of our knowledge this is the strongest existing positive result in this direction.

Remark 3.4. The above proof depends on the fact that a group is a very symmetric object, and thus there are generally many directions to escape to infinity. This is certainly not the case for more general bounded geometry metric spaces or manifolds, as the following example shows. Consider the set $X = \mathbb{N} \times \{0\} \cup \left(\bigcup_{i \in \mathbb{N}} \{n\} \times \{0, 1, \ldots, a_n\}\right) \subseteq \mathbb{Z} \times \mathbb{Z}$, where a_n is a sequence of natural numbers which increases to infinity. The metric on X (the "lumberjack metric") is defined by

$$d(\,(x, y), (x', y')\,) = |y| + |x - x'| + |y'|.$$

In this metric space there is only one way to infinity, that is via the horizontal line, hence the "tails" that constitute a cycle which kills the fundamental class have to escape through this one route. By controlling the growth of the sequence a_n one can easily impose any lower bound on the growth of any 1-chain which kills the fundamental class $[X]$.

4. $H_0^f(\Gamma)$ AND ISOPERIMETRIC INEQUALITIES

There is number of isoperimetric inequalities which one can study on a finitely generated group on a bounded geometry space X. The one we are interested in, studied in [9, 29], is the isoperimetric inequality of the form

$$\#A \leq C \sum_{x \in \partial A} f(|x|) \qquad (I_\partial^f)$$

for all finite sets $A \subset X$, where ∂A is the boundary of A (see below). We will say that X *satisfies inequality* (I_∂^f) if there exists a constant $C > 0$ such that (I_∂^f) holds for every finite set $A \subseteq X$. When f is constant this isoperimetric inequality is equivalent to non-amenability of the group.

We set

$$\partial A = \{g \in \Gamma : \ d(g,A) = 1 \text{ or } d(g, \Gamma \setminus A) = 1\},$$
$$\partial^{\mathrm{e}} A = \{(g,h) \in A \times (\Gamma \setminus A) : \ d(g,h) = 1\} \cup \{(g,h) \in (\Gamma \setminus A) \times A : \ d(g,h) = 1\}.$$

The first lemma reformulates the inequality (I_∂^f) in terms of functions. This form will be much more convenient to work with in connection to coarse homology. Recall that for an n-simplex $\overline{x}$ we denote by $|\overline{x}|$ the distance of $\overline{x}$ from the fixed point $\overline{e}$. In particular, if the simplex is an edge (x,y), we use the notation $|(x,y)|$.

Lemma 4.1. *Let X be a metric space of bounded geometry. The following two conditions are equivalent*

(a) *X satisfies inequality (I_∂^f),*

(b) *the inequality*

$$(I_\nabla^f) \qquad \sum_{x \in X} |\eta(x)| \ \le \ D \left(\sum_{x\in X} \sum_{y \in B(x,1)} |\eta(x) - \eta(y)| \, f(|(x,y)|) \right)$$

holds for every finitely supported function $\eta : X \to \mathbb{R}$ and a constant $D > 0$.

Proof. (b) implies (a) follows just by applying the inequality to $\eta = 1_A$, and using the property (f_1). To prove the other direction we use a standard "co-area" argument. It is enough to restrict to the case $\sum_{x\in X} |\eta(x)| = 1$ and we first consider $\eta \ge 0$. By density arguments it is sufficient prove the claim for functions of ℓ_1-norm 1 which take values in sets of the form $\left\{\frac{i}{M}\right\}_{i\in\mathbb{N}}$ (where M is chosen for each function separately). For such η we denote $A^{(i)} = \left\{x \in X : \ \eta(x) > \frac{i}{M}\right\}$. Then we can write $\eta(x) = \frac{1}{M} \sum_{i\in\mathbb{N}} 1_{A^{(i)}}(x)$ and $\sum_{i\in\mathbb{N}} \# A^{(i)} = M$. Furthermore,

$$\begin{aligned}
\sum_{d(x,y)\le 1} |\eta(x) - \eta(y)| \, f(|(x,y)|) &= \frac{1}{M} \sum_{i\in\mathbb{N}} \left(\sum_{x\in X} \sum_{y\in B(x,1)} \left| 1_{A^{(i)}}(x) - 1_{A^{(i)}}(y) \right| f(|(x,y)|) \right) \\
&\ge \sum_{i\in\mathbb{N}} \frac{1}{M} \left(\sum_{(x,y)\in\partial^e A^{(i)}} f(|(x,y)|) \right) \\
&\ge C \sum_{i\in\mathbb{N}} \frac{1}{M} \left(\sum_{x\in\partial A^{(i)}} f(|x|) \right) \\
&\ge \sum_{i\in\mathbb{N}} \frac{1}{M} \# A^{(i)} \\
&= 1.
\end{aligned}$$

For a general η apply the above inequality to $|\eta|$ together with the triangle inequality to obtain

$$\sum_{x\in X}|\eta(x)| \leq \sum_{d(x,y)\leq 1}\Big||\eta(x)|-|\eta(y)|\Big|f(|(x,y)|) \leq \sum_{d(x,y)\leq 1}|\eta(x)-\eta(y)|f(|(x,y)|).$$

□

The next theorem gives a homological description of the inequality (I_∂^f) and is the second of the main results of this paper. It can be understood as providing a passage from knowing some information "on average" (the inequality) to precise, even distribution of coefficients of $\psi \in C_1^f(X)$ which satisfies $\partial\psi = [X]$.

Theorem 4.2. *Let X be a bounded geometry quasi-geodesic metric space. The following conditions are equivalent:*

(A) *X satisfies inequality (I_∂^f),*
(B) *$[X] = 0$ in $H_0^f(X)$.*

Some comments are in order before we prove the theorem. In the uniformly finite case [3] the proof of one implication relies on the Hahn–Banach theorem, which allows to build a functional distinguishing between boundaries and the fundamental class. However, our chain groups have a topology which does not allow to conveniently generalize this argument. There is another way to view the equivalence *X non-amenable if and only if* $H_0^{\mathrm{uf}}(X) = 0$, which we roughly sketch here (with the notation from the proof of Theorem 4.2 below).

Non-amenability of X is equivalent to the inequality $\|\eta\|_1 \leq C\|\delta\eta\|_1$, where $\delta : \ell_1(X) \to \ell_1^{\mathrm{ch}}(N\Delta)$ is defined by $\delta\eta(x,y) = \eta(y) - \eta(x)$, $N\Delta$ is the 1-neighborhood of the diagonal in $X\times X$ and $\ell_1^{\mathrm{ch}}(N\Delta)$ denotes the absolutely summable 1-chains of propagation at most 1. As a linear map between Banach spaces δ is continuous and, by the above inequality, also topologically injective. Since the topological dual of $\ell_1^{\mathrm{ch}}(N\Delta)$ is essentially $C_1^{\mathrm{uf}}(X)$, this is further equivalent, by duality, to surjectivity of the adjoint map $\widetilde{\partial} : C_1^{\mathrm{uf}}(X) \to C_0^{\mathrm{uf}}(X)$ (the latter space can be simply viewed as $\ell_\infty(X)$) which is, up to a multiplicative constant, the same as our differential. This is however exactly the vanishing of the 0-dimensional homology group.

It is this point of view which we use to prove Theorem 4.2. However, if f is not constant, the maps ∂ and δ are not continuous and thus do not respond directly to the above argument. Before embarking on the proof, we record a simple but necessary fact.

Lemma 4.3. *Let X be a bounded geometry quasi-geodesic metric space. Then $[X] = 0$ in $H_0^f(X)$ if and only if $1_X = \partial\psi$, where $\psi \in C_1^f(X)$ and $\mathscr{P}(\psi) \leq 1$.*

Proof. To prove the nontrivial direction replace every edge $[x,y]$ with $d(x,y) > 1$ and a non-zero coefficient $a(x,y)$ by the chain $a(x,y)\sum[x_i,x_{i+1}]$ where the x_i are given by the quasi geodesic condition, so that $d(x_i,x_{i+1}) \leq 1$. □

Proof of Theorem 4.2. We equip $X\times X$ with the measure $\nu(x,y) = f(|(x,y)|)$. Let $N\Delta = \{(x,y)\in X\times X : d(x,y)\leq 1\}$ denote the 1-neighborhood of the diagonal.

We consider the linear space

$$\ell_\infty^f(N\Delta) = \left\{ \psi : N\Delta \to \mathbb{R} : \sup_{(x,y)\in N\Delta} \frac{|\psi(x,y)|}{f(|(x,y)|)} < \infty \right\}$$

with the norm

$$\|\psi\|_\infty^f = \sup_{(x,y)\in N\Delta} \frac{|\psi(x,y)|}{f(|(x,y)|)}.$$

Denote also

$$\ell_1(N\Delta,\nu) = \left\{ \psi : N\Delta \to \mathbb{R} : \sum_{(x,y)\in N\Delta} |\psi(x,y)|\nu(x,y) < \infty \right\}$$

and equip it with the norm

$$\|\psi\|_{1,\nu} = \sum_{(x,y)\in N\Delta} |\psi(x,y)|\nu(x,y).$$

For $\psi,\phi : X^n \to \mathbb{R}$, we denote

$$\langle \psi,\phi\rangle = \sum_{\overline{x}\in X^{n+1}} \psi(\overline{x})\phi(\overline{x}),$$

whenever this expression makes sense. The topological dual of $\ell_1(N\Delta,\nu)$ with respect to this pairing is $\ell_\infty^f(N\Delta)$. Let $\mathbb{F}$ denote finitely supported functions on X and define $\delta : \mathbb{F} \to \ell_1(N\Delta,\nu)$ to be the map defined by

$$\delta\eta(x,y) = \eta(y) - \eta(x), \quad \text{for } d(x,y) \le 1.$$

Then define a linear operator $\widetilde{\partial} : \ell_\infty^f(N\Delta) \to \ell_\infty^f(X)$ by setting

$$\widetilde{\partial}\psi(x) = \sum_{y\in B(x,1)} \psi(y,x) - \psi(x,y).$$

On chains, $\widetilde{\partial}$ is algebraically dual to δ with the above pairings, i.e.

$$\langle \eta, \widetilde{\partial}\psi\rangle = \langle \delta\eta,\psi\rangle$$

as finite sums for all $\eta \in \mathbb{F}$ and $\psi \in C_1^f(X)$ such that $\mathscr{P}(\psi) \le 1$. Note that $\widetilde{\partial}$ is also a linear extension of 2∂ from $C_1^f(X)$ to $\ell_\infty^f(N\Delta)$ for such ψ.

(B) $\Longrightarrow$ (A). Assume that there exists a chain $\psi \in C_1^f(X)$ such that $1_X = \partial\psi$ and $\mathscr{P}(\psi) = 1$ (by Lemma 4.3). For a non-negative function $\eta \in \mathbb{F}$ we have

$$\begin{aligned}
\|\eta\|_1 &= \sum_{x\in X} \eta(x) \\
&= \sum_{x\in X} \eta(x)\cdot \partial\psi(x) \\
&= \frac{1}{2} \sum_{(x,y)\in X\times X} \delta\eta(x,y)\cdot\psi(x,y) \\
&\leq \frac{1}{2} \sum_{d(x,y)\leq 1} |\eta(x)-\eta(y)|\; |\psi(x,y)| \\
&\leq C \sum_{d(x,y)\leq 1} |\eta(x)-\eta(y)|\; f(|(x,y)|).
\end{aligned}$$

By Lemma 4.1, we are done.

(A) $\Longrightarrow$ (B). By Lemma 4.3 there is no loss of generality by restricting to functions of propagation 1. Let $\delta\mathbb{F}$ denote the image of $\mathbb{F}$ under δ. By Lemma 4.1 we rewrite the inequality (I^f_∂) in the functional form as (I^f_∇) i.e.,

$$\sum |\eta(x)| \leq C\left(\sum_{d(x,y)\leq 1} |\eta(x)-\eta(y)| f(|(x,y)|)\right)$$

for every $\eta \in \mathbb{F}$. If we interpret this inequality in terms of the norms of elements of $\mathbb{F}$ and $\delta\mathbb{F}$ it reads $\|\eta\|_1 \leq C\|\delta\eta\|_{1,\nu}$. The map $\delta : \mathbb{F} \to \delta\mathbb{F}$ is a bijection, thus there exists an inverse $\delta^{-1} : \delta\mathbb{F} \to \mathbb{F}$. The inequality implies that this inverse is continuous. We extend it to a continuous map

$$\delta^{-1} : \overline{\delta\mathbb{F}} \to \ell_1(X),$$

where $\overline{\delta\mathbb{F}}$ is the norm closure of $\delta\mathbb{F}$ in $\ell_1(N\Delta,\nu)$. This map induces a continuous adjoint map $(\delta^{-1})^* : C_0^{\mathrm{uf}}(X) \to \overline{\delta\mathbb{F}}^*$ between the dual spaces, which satisfies

$$\left\langle \delta\eta, (\delta^{-1})^*\zeta \right\rangle = \langle \eta, \zeta\rangle$$

for $\eta \in \mathbb{F}$ and $\zeta \in C_0^{\mathrm{uf}}(X)$. We thus have the following diagrams, where the top one is dual to the bottom one:

$$\begin{tikzcd}
\ell_\infty^f(N\Delta) \arrow[d, "i^*"'] & \\
\overline{\delta\mathbb{F}}^* & C_0^{\mathrm{uf}}(X) \arrow[l, "(\delta^{-1})^*"']
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{\delta\mathbb{F}} \arrow[r, shift left, "\delta^{-1}"] \arrow[d, hook, "i"'] & \ell_1(X) \arrow[l, dashed, shift left, "\delta"] \\
\ell_1(N\Delta, \nu) &
\end{tikzcd}$$

Here i is the natural injection, i^* is the restriction and the dashed arrow denotes a discontinuous map which is densely defined on $\ell_1(X)$.

Note now that if $\eta \in \mathbb{F}$ and $\psi \in \ell^f_\infty(X)$ is a 1-chain, the duality $\langle \widetilde{\partial}\psi, \eta \rangle = \langle \psi, \delta\eta \rangle$ says that ψ is determined on $\delta\mathbb{F}$ only. This allows to construct a preimage of the fundamental class in the following way. Take ϕ to be any element in $\ell^f_\infty(N\Delta)$ such that
$$i^*\phi = (\delta^{-1})^* 1_X,$$
(for instance any extension of $(\delta^{-1})^* 1_X$ to a functional on the whole $\ell_1(N\Delta; \nu)$ guaranteed by the Hahn-Banach theorem). Then ϕ is an element of $\ell^f_\infty(N\Delta)$ and might not a priori belong to $C^f_1(X)$. To correct this we anti-symmetrize ϕ. Denote the transposition $\phi^T(x, y) = \phi(y, x)$. We define
$$\psi = \phi - \phi^T.$$
Then $\psi^T = -\psi$ so that $\psi \in C^f_1(X)$ and we will now show that $\partial\psi = 1_X$ in $C^f_0(X)$. For every $x \in X$ and its characteristic function 1_x we have

$$\begin{aligned}
\widetilde{\partial}\psi(x) &= \left\langle 1_x, \widetilde{\partial}\psi \right\rangle \\
&= \langle \delta 1_x, \psi \rangle \\
&= \langle \delta 1_x, \phi \rangle - \left\langle \delta 1_x, \phi^T \right\rangle \\
&= \langle \delta 1_x, \phi \rangle + \left\langle \delta 1^T_x, \phi^T \right\rangle \\
&= 2 \langle \delta 1_x, \phi \rangle \\
&= 2 \langle \delta 1_x, i^*\phi \rangle \\
&= 2 \left\langle \delta 1_x, (\delta^{-1})^* 1_X \right\rangle \\
&= 2 \langle 1_x, 1_X \rangle \\
&= 2.
\end{aligned}$$

In the above calculations we used the fact that $\delta 1^T_x = -\delta 1_x$ and that $\langle \psi, \phi \rangle = \langle \psi^T, \phi^T \rangle$. Finally since $2\partial = \widetilde{\partial}$ so $\partial\psi = 1_X$. □

It is interesting to note that the linear space of 1-chains of propagation at most 1 is a complemented subspace of $\ell^f_\infty(N\Delta)$. Indeed, the anti-symmetrizing map $P(\phi) =$

$\phi - \phi^T$ is a bounded projection from $\ell_\infty^f(N\Delta)$ onto that subspace. Consequently our argument shows that actually for any $\eta \in C_0^{\mathrm{uf}}(X) = \ell_\infty(X)$, and a lifting $\phi_\eta \in \ell_\infty^f(N\Delta)$ of $(\delta^{-1})^*\eta$, the projection $P\phi_\eta$ satisfies $\partial P\phi_\eta = \eta$, so that P and $(\delta^{-1})^*$ together with the Hahn-Banach theorem are used to construct a right inverse to $\widetilde{\partial}$. In other words, we get the following

Corollary 4.4. *For a bounded geometry metric space X the following are equivalent:*

1. $[X] = 0$ *in* $H_0^f(X)$,
2. *the homomorphism* $j_0^f : H_0^{\mathrm{uf}}(X) \to H_0^f(X)$, *induced by inclusion of chains, is trivial.*

The above fact was known in the uniformly finite case [3, Proposition 2.3]. Another corollary of the proof is a different proof, via the isoperimetric inequality, of Lemma 2.4. We can also identify $\overline{\delta\mathbb{F}}^*$ with $\ell_\infty^f(N\Delta)/\operatorname{Ann}(\overline{\delta\mathbb{F}})$ up to an isometric isomorphism, where $\operatorname{Ann}(E) = \{\psi \in E^* : \langle\psi, \eta\rangle = 0 \text{ for } \eta \in E\}$ is the annihilator of E.

Remark 4.5. Note that if $f \equiv$ const and we replace ℓ_1 and ℓ_∞ by ℓ_p and ℓ_q respectively ($1 < p, q < \infty$, $1/p + 1/q = 1$), then the above argument together with the fact that non-amenability is equivalent to $\|\eta\|_q \le C\|\delta\eta\|_q$ for any $1 \le q < \infty$, gives a different proof of a theorem in [27] and [8], namely that vanishing of the 0-th ℓ_q-homology group characterizes non-amenability of metric spaces.

Remark 4.6. The above proof gives a dual characterization of the best constant C in the isoperimetric inequality in terms of the distance from the origin to the affine subspace $\partial^{-1}(1_X)$ in $\ell_\infty^f(X)$. In particular for $f \equiv 1$ this applies to the Cheeger constant of X.

Remark 4.7. Combining the result of Żuk [29, Theorem 1] with Theorem 4.2 gives another proof of the Theorem 3.1, but of course the proof in the previous section has the advantage of being constructive while the above is only an existence statement.

5. Examples

Wreath products. In [29] Żuk proved the inequality (I_∂^f) with $f(t) = t$ on any finitely generated group and asked if there are groups for which f can be chosen to be of slower growth. Examples of such groups were constructed by Erschler [9]. Recall that the (restricted) wreath product is defined as the semidirect product

$$G \wr H = \left(\bigoplus_{h\in H} G\right) \rtimes H$$

where H acts on $\bigoplus_{h\in H} G$ by translation of coordinates. Using Erschler's results, together with the above characterization, we exhibit finitely generated groups for which $[\Gamma] = 0$ in $H_0^f(\Gamma)$ with f growing strictly slower than linearly. In [9] Erschler showed that for a group of the form $F \wr \mathbb{Z}^d$, where $d \ge 2$ and F is a non-trivial finite group, inequality (I_∂^f) holds with $f(n) = n^{1/d}$. This gives

Corollary 5.1. *Let $\Gamma = F \wr \mathbb{Z}^d$ for a non-trivial finite group F. Then $[\Gamma] = 0$ in $H_0^f(\Gamma)$, where $f(n) = n^{1/d}$.*

It is also possible to exhibit groups for which the fundamental class vanishes in $H_0^f(\Gamma)$ for $f(n) = \ln n$.

Corollary 5.2. *Let $\Gamma = F \wr (F \wr \mathbb{Z})$ for a non-trivial finite group F. Then $[\Gamma] = 0$ in $H_0^f(\Gamma)$, where $f(n) = \ln n$.*

Iteration of the wreath product leads to successively slower growing functions, see [9].

Polycyclic groups. Let us denote

$$\operatorname{rad} F = \min \{r : \ F \subseteq B(g, r), g \in \Gamma\}.$$

Definition 5.3. *Let Γ be an infinite, amenable group. We define the isodiametric profile $D_\Gamma : \mathbb{N} \to \mathbb{N}$ of Γ by the formula*

$$D_\Gamma(r) = \sup \frac{\#F}{\#\partial F},$$

where the supremum is taken over all finite sets $F \subset \Gamma$ with the property $\operatorname{rad} F \leq r$.

In other words, the isodiametric profile finds finite sets with the smallest boundary among those with prescribed diameter. It can be equivalently described as the smallest function D such that the inequality $\dfrac{\#F}{\#\partial F} \leq D(\operatorname{rad} F)$ holds for all finite subsets $F \subseteq \Gamma$. It is easy to see that D is sublinear, in the sense that $D(n) \leq Cn$ for some $C > 0$ and all $n > 0$. Also, in some sense, D is an inverse of the function A_X introduced in [14].

Proposition 5.4. *Let Γ be a finitely generated group. If Γ satisfies inequality (I_∂^f) then there exists $C > 0$ such that*

$$D \leq Cf.$$

In particular, the above estimate holds when the fundamental class of $[\Gamma]$ vanishes in $H_0^f(\Gamma)$.

Proof. Let $F \subset \Gamma$ be a finite subset. Translating to the origin we can assume that $F \subseteq B(e, \operatorname{rad} F)$. By inequality ($I_\partial^f$) and monotonicity of f we obtain

$$\#F \leq C \sum_{x \in \partial F} f(|x|) \leq \#\partial F \cdot Cf(\operatorname{rad} F),$$

which proves the claim. □

It is well-known that for infinite polycyclic groups D grows linearly [16]. Thus we have

Corollary 5.5. *Let Γ be an infinite, polycyclic group. Then the fundamental class $[\Gamma]$ vanishes in $H_0^f(\Gamma)$ if and only if f is linear.*

6. Obstructions to weighted Poincaré inequalities

In [13] the authors studied a weighted Poincaré inequality of the form

$$(P_\rho) \qquad \int_M \eta(x)^2\rho(x)\, v \;\leq\; C \int_M |\nabla\eta|^2\, v$$

(v is the volume form) for $\rho > 0$ and its applications to rigidity of manifolds. For the purposes of [13] it is useful to know what ρ one can choose since the rigidity theorems of that paper hold under the assumption that the curvature is bounded below by ρ. Just as with f, we assume throughout that ρ is a function of the distance from a base point and that $\rho > 0$.

In this section we establish a relation between the inequalities (I_∂^f) and (P_ρ). First we need an auxiliary notion. Let $f : [0,\infty) \to [0,\infty)$ be non-decreasing. We will say that f *is a slowly growing function* if for every $\varepsilon > 0$ there is a t_0 such that for every $t > t_0$ we have $f(t+1) \leq f(t) + \varepsilon$. Examples of such f are all convex sublinear functions, for instance $f(t) = t^\alpha$ for $\alpha < 1$ and $f(t) = \ln t$. A sufficient condition is that f is convex and $\lim_{t\to\infty} f'(t) = 0$. The main result of this section is the following.

Theorem 6.1. *Let $f : [0,\infty) \to [0,\infty)$ be a non-decreasing, slowly growing function. Let M be a compact manifold such that the universal cover satisfies the weighted Poincaré inequality (P_ρ) with weight $\rho : \widetilde{M} \to \mathbb{R}$ given by $\rho(x) = 1/f(d(x,x_0))$ for every smooth compactly supported function $\eta : \widetilde{M} \to \mathbb{R}$ and a fixed point $x_0 \in \widetilde{M}$. Then $[\Gamma] = 0$ in $H_0^f(\Gamma)$ where $\Gamma = \pi_1(M)$.*

To prove Theorem 6.1 we will use techniques from a classical paper of Brooks [5]. We first consider a fundamental domain F for the action of the fundamental group on $\widetilde{M}$ by taking a smooth triangulation and choosing for each n-simplex Δ in M a simplex $\widetilde{\Delta}$ in $\widetilde{M}$, which covers Δ. The fundamental domain is the union of these simplices.

Lemma 6.2. *Let $\rho : \widetilde{M} \to \mathbb{R}$ be constant when restricted to γF for every $\gamma \in \Gamma$. If the continuous weighted Poincaré inequality (P_ρ) with weight ρ holds for every compactly supported smooth function $\eta : \widetilde{M} \to \mathbb{R}$ then the isoperimetric inequality*

$$(I_\rho) \qquad \sum_{x\in A} \rho_d(x) \leq C\#\partial A$$

holds for every finite subset $A \subset \Gamma$ where $\rho_d(x) = \rho(xF)$.

Proof. Assume the contrary. Then there is a sequence of sets $A_i \subset \Gamma$ such that

$$\frac{\#\partial A_i}{\sum_{x\in A_i} \rho_d(x)} \;\longrightarrow\; 0.$$

As in [6] we can construct smooth functions on the universal cover, which exhibit the same asymptotic behavior. Choose $0 < \varepsilon_0 < \varepsilon_1 < 1$, where both ε_i are sufficiently small, and a function $\kappa : [0,1] \to [0,1]$ such that $\kappa(t) = 1$ when $t \geq \varepsilon_1$ and $\kappa(t) = 0$ when $t \leq \varepsilon_0$.

Let χ_i be the characteristic function of $B_i = \bigcup_{\gamma\in A_i}\gamma F$ and let $\eta_i = \chi_i\,\kappa(d(x, \widetilde{M}\setminus B_i))$. Then one can estimate

$$\int |\nabla\eta_i|^2 \leq C\mathrm{Vol}(\partial B_i) \leq C'\#\partial A_i.$$

On the other hand

$$\int \eta_i^2(x)\rho(x) \geq D\sum_{\gamma\in A_i}\int_{\gamma F}\rho(x) \geq D\,\mathrm{Vol}(F)\sum_{\gamma\in A_i}\rho_d(\gamma).$$

Consequently

$$\frac{\int|\nabla\eta|^2}{\int\eta(x)^2\rho(x)} \quad\longrightarrow\quad 0$$

and we reach a contradiction. □

We also need an unpublished theorem of Block and Weinberger (see [27] for a proof).

Theorem 6.3 (Block-Weinberger, Whyte)**.** *Let $c = \sum_{x\in X} c_x[x] \in C_0^{\mathrm{uf}}(X)$. Then $[c] = 0$ in $H_0^{\mathrm{uf}}(X)$ if and only if the inequality*

$$\left|\sum_{x\in A} c_x\right| \leq C\#\partial A$$

holds for every finite $A \subset \Gamma$ and some $C > 0$.

We observe that by inequality (I_ρ) and the above theorem the class represented by $c = \sum_{x\in X}\rho_d(x)[x] \in C_0^{\mathrm{uf}}(X)$ vanishes in the uniformly finite homology group. We will denote by $[1/f]$ the homology class represented by the chain $\sum_{x\in X} 1/f(|x|)$. Theorem 6.1 now follows from the following

Lemma 6.4. *Let f be non-decreasing slowly growing function. If $[1/f] = 0$ in $H_0^{\mathrm{uf}}(X)$ then $[X] = 0$ in $H_0^f(X)$.*

Proof. Let $1/f(|x|) = \partial\phi(x)$, where $\phi \in C_0^{uf}(X)$ and $\mathscr{P}(\phi) = 1$. Let $\psi(x,y) = \phi(x,y)f(|x,y|)$. Then $\psi \in C_1^f(X)$ and

$$\begin{aligned}\partial\psi(x) &= \sum_{y\in B(x,1)}\phi(y,x)f(|(x,y)|)\\ &= \sum_{y\,(+)}\phi(y,x)f(|(x,y)|) - \sum_{y\,(-)}\phi(x,y)f(|(x,y)|)\end{aligned}$$

where $(+)$ and $(-)$ denote sums over edges (y,x) with positive coefficients and negative coefficients respectively. Continuing we have

$$\geq \sum_{y\,(+)}\phi(y,x)f(|x|) - \sum_{y\,(-)}\phi(x,y)f(|x|+1)$$

Let $\mathcal{N}$ be such that $\#B(x,1) \leq \mathcal{N}$ for every $x \in X$. Fix $0 < C < 1$ and let $\varepsilon > 0$ be such that $1 - \varepsilon\|\phi\|_\infty\,\mathcal{N} > C$. By assumption there exists a compact set $K \subset X$ such that outside of K we have $f(|x|+1) \leq f(|x|)+\varepsilon$. By factoring $f(|x|)$ from first two sums we have

$$\begin{aligned}
&= f(|x|)\left(\sum_{y\in B(x,1)} \phi(y,x)\right) - \varepsilon\left(\sum_{y\,(-)} \phi(x,y)\right)\\
&\geq \partial\phi(x) f(|x|) - \varepsilon\left(\sum_{y\,(-)} \phi(x,y)\right)\\
&\geq 1-\varepsilon\|\phi\|_\infty \mathcal{N}\\
&\geq C
\end{aligned}$$

for every $x \in X \setminus K$. To achieve the same lower bound on K we can add finitely many tails t_x to ψ to assure $\partial\psi(x) \geq C$ on K. We do not alter any of the previously ensured properties since K is finite. The 1-chain ψ' that we obtain in this way satisfies $\psi' \in C_1^f(X)$ and $\partial\psi'(x) \geq C$, thus by Lemma 2.4 the fundamental class vanishes as well. □

Theorem 6.1 can be generalized to open manifolds under suitable assumptions. We leave the details to the reader.

Remark 6.5. We would like to point out that it is not hard to show that the isoperimetric inequality (I_∂^f) is implied by a discrete inequality

$$\sum_{x\in X} |\eta(x)|^2 \leq C \sum_{d(x,y)\leq 1} |\eta(x)-\eta(y)|^2 f(|(x,y)|)$$

for all finitely supported $\eta : X \to \mathbb{R}$. The latter can be interpreted as the existence of a spectral gap for a weighted discrete Laplace operator.

7. Primitives of differential forms

Sullivan in [24] studied the growth of a primitive of a differential form and asked a question about the connection of a certain isoperimetric inequality and the existence of a bounded primitive of the volume form on a non-compact manifold. This question was later answered by Gromov [10] and other proofs are provided by Brooks [6] and Block and Weinberger [3]. For unbounded primitives this question was studied by Sikorav [25], and Żuk [29].

As an application of the controlled coarse homology we obtain precise estimates on growth of primitives of the volume form on covers of compact Riemannian manifolds.

Theorem 7.1. *Let N be an open, complete Riemannian manifold of bounded geometry and let $X \subset N$ be a discrete subset of N which is quasi-isometric to N. Then $[X] = 0$ in $H_0^f(X)$ if and only if the volume form on N has a primitive of growth controlled by f.*

Proof. First note that by quasi-isometry invariance of the controlled homology (corollary 2.3), it is sufficient to prove the statement for any X that is quasi-isometric to N.

We follow Whyte's proof [28, Lemma 2.2.]. Let $\kappa > 0$ be smaller than the convexity radius of N. We choose a maximal κ-separated subset $X \subset N$ and consider the partition of unity $\{\varphi_x\}_{x \in X}$ associated to the cover by balls centered at points of X of radius κ. Denote $v_x = \varphi_x v$.

Given $\psi \in C_1^f(X)$ of propagation $\mathscr{P}(\psi) \leq r$, which bounds the fundamental class we note that $v_x - v_y = d\omega_{(x,y)}$, where $\omega_{(x,y)}$ are $(n-1)$-forms of uniformly bounded supports. If we let $\omega = \sum_{d(x,y)\leq r} \psi(x,y)\omega_{(x,y)}$ then we have

$$d\omega = \sum_{d(x,y)\leq r} \psi(x,y)(v_x - v_y) = \sum_{x \in X} \widetilde{\partial}\psi_x v_x = 2v.$$

On the other hand, if $v = d\omega$ and $|\omega| \leq Cf$ then it follows from Stokes' theorem that the isoperimetric inequality with f holds for X. By Theorem 4.2 this implies vanishing of the fundamental class in $H_0^f(X)$. $\square$

The above gives a different proof of a theorem of Sikorav [25] and also explains the nature of the extra constants appearing in the formulation of the main theorem of that paper. These constants reflect the fact that the growth of the primitive of a differential form is of large-scale geometric nature.

For the rest of this section, we fix the notation as follows: M is a compact manifold with a universal cover $\widetilde{M}$ and fundamental group $\pi_1(M)$.

Corollary 7.2. *Let M be a compact manifold with $\Gamma = \pi_1(M)$. Then $[\Gamma] = 0$ in $H_0^f(\Gamma)$ if and only if the volume form on the universal cover $\widetilde{M}$ has a primitive whose growth is controlled by f.*

For various classes of groups we obtain specific estimates based on the results from previous sections. For instance for polycyclic groups we have the following

Corollary 7.3. *Let $\pi_1(M)$ be an infinite, polycyclic group. Then the primitive of the volume form on $\widetilde{M}$ has exactly linear growth.*

For groups with finite asymptotic dimension the dichotomy between amenable and non-amenable groups is manifested in a gap between possible growth type of the primitives of volume forms. We discuss only the Baumslag-Solitar groups

$$\mathrm{BS}(m,n) = \langle a, b \mid ab^m a^{-1} = b^n \rangle,$$

which have finite asymptotic dimension of linear type and are either solvable or have a non-abelian free subgroup.

Corollary 7.4. *Let M be a compact manifold with $\pi_1(M) = \mathrm{BS}(m,n)$. Then a primitive of the volume form on $\widetilde{M}$ has*

(1) *bounded growth if $\pi_1(M)$ is non-amenable i.e. $|m| \neq 1 \neq |n|$,*
(2) *linear growth if $\pi_1(M)$ is amenable i.e., $|m| = 1$ or $|n| = 1$.*

8. Other applications and final remarks

8.1. **Distortion of subgroups.** Let $G \subseteq \Gamma$ be a subgroup. Then the inclusion is always a coarse embedding and it is often a question whether G is undistorted in Γ, that is the inclusion is a quasi-isometric embedding. We have the following

Proposition 8.1. *Let $G \subseteq \Gamma$ be a inclusion of a subgroup with both G and Γ finitely generated. If $[G] = 0$ in $H_0^f(G)$ then $[\Gamma] = 0$ in $H_0^{f\circ\rho_-^{-1}}(\Gamma)$.*

Proof. It is straightforward to check that the pushforward of a 1-chain that bounds the fundamental class $[G]$ in $H_0^f(G)$ will have growth controlled by $f \circ \rho_-^{-1}$ on Γ. We then partition Γ into G-cosets, and construct a 1-chain that bounds $[\Gamma]$ as the sum of translates of the chain that bounds $[G]$. □

In particular if G is undistorted in Γ, then the fundamental classes should vanish in appropriate groups with the same control function f. This fact can be used to estimate the distortion of a subgroup or at least to decide whether a given subgroup can be an undistorted subgroup of another group, however except the iterated wreath products mentioned earlier we do not know examples in which explicit computation would be possible.

8.2. **Questions.** A natural question is whether one can introduce a homology or cohomology theory which would in a similar way reflect isoperimetric properties of amenable actions. Isoperimetric inequalities for such actions are discussed in [15].

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