Property (T) for $\mathrm{Aut}(F_n)$

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Property (T) was introduced by Kazhdan in 1966 and has become a fundamental rigidity property for groups. Only a few classes infinite groups are known to satisfy property (T), including most notably lattices in higher rank Lie groups, automorphism groups of thick buildings and certain random hyperbolic groups. We refer to [1] as the most comprehensive overview of property (T).

Recently a new characterization of property (T) due to Ozawa [8] described property (T) for a finitely generated group G in terms of an algebraic spectral gap-type condition, expressed in the form of the equation

$$\Delta^2 - \lambda\Delta = \sum_{i=1}^{k} \xi_i^* \xi_i, \tag{1}$$

in the real group ring $\mathbb{R}G$, where $\Delta = 1 - |S|^{-1} \sum_{s\in S} s$ is the Laplacian associated to a finite, symmetric generating set S, $\lambda > 0$ and $\xi_i \in \mathbb{R}G$. This characterization allowed for a new strategy to be used to prove property (T) for infinite groups: solving (1) with the aid of positive definite programming. This method was first implemented by Netzer and Thom [7] to give a new proof of property (T) for the group $\mathrm{SL}_3(\mathbb{Z})$ and to improve significantly the estimate of its Kazhdan constant. Similar result were later obtained for $\mathrm{SL}_n(\mathbb{Z})$ by Fujiwara and Kabaya for $n = 3, 4$ [2] and Kaluba and Nowak for $n = 3, 4, 5$ [3].

The first new group for which property (T) was proved using this new approach was $\mathrm{Aut}(F_5)$, the automorphism group of the free group on 5 generators [4]. The infinite case was settled subsequently by Kaluba, Kielak and Nowak [5] in the form of the following

Theorem 1. *The group* $\mathrm{Aut}(\mathrm{F_n})$ *has property* (T) *for* $n \geq 6$.

As numerical methods can only be applied to a single group at a time, a key ingredient of the proof of the above theorem is a technique of decomposing equation (1) in the group $\mathrm{SAut}(\mathrm{F_n})$, a subgroup of $\mathrm{Aut}(F_n)$ of index 2, into smaller pieces. More precisely, whenever $m \geq n$ the Laplacian element $\Delta_m \in \mathbb{R}\,\mathrm{SAut}(F_m)$ can be written in terms of copies of the Laplacian $\Delta_n \in \mathrm{SAut}(F_n)$, that are stitched together by the action of the alternating group Alt_m.

A similar idea is then applied to the square of the Laplacian $\Delta^2 \in \mathrm{SAut}(F_m)$, which is first decomposed into a sum of three other elements of the group ring, the square part Sq_m, the opposite part Op_m and the adjacent part Adj_m. For each of these elements there is a separate decomposition formula, which allows to express such an element in $\mathbb{R}\,\mathrm{SAut}(F_m)$ in terms of sums of translates of corresponding elements $\mathbb{R}\,\mathrm{SAut}(F_n)$. These facts allow to express the equation $\Delta_m^2 - \lambda\Delta_m$, for a certain positive λ, in terms of the operators Sq_n, Op_n and Adj_n. Then a single computation using semidefinite programming in the ring $\mathbb{R}\,\mathrm{SAut}(F_5)$ is used to finish the proof.

The same argument applies in the case of the family $\mathrm{SL}_n(\mathbb{Z})$, where an algebraic spectral gap certified in the group ring of $\mathrm{SL}_5(\mathbb{Z})$ additionally yields new estimates on Kazhdan constants of $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 6$, which asymptotically are $1/2$ of the well-known upper bound of $\sqrt{\frac{2}{n}}$.

Another consequence is a new answer to a question of Lubotzky about the dependence on the generating of the property of being expanders. Indeed, as $\mathrm{Aut}(F_n)$, $n \geq 3$, are residually alternating, there exists a sequence of alternating groups Alt_{k_i} such that as quotients of $\mathrm{Aut}(F_n)$ and with the inherited generating set their Cayley graphs form a sequence of expanders, while they are known not to be expanders with with respect to their usual generating sets.

Another application concerns an algorithm used to generate random elements in finite groups. Lubotzky and Pak [6] studied the Product Replacement Algorithm and have observed that it can be described in terms of a certain natural action of $\mathrm{Aut}(F_n)$ and the associated random walk, whose fast convergence would be implied by a spectral gap. Our results showing property (T) for $\mathrm{Aut}(F_n)$ for $n \geq 5$, combined with those of Lubotzky and Pak, thus provide the explanation of the surprisingly fast convergence of the Product Replacement Algorithm.

References

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